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Satellite-to-satellite tracking and satellite gravity gradiometry (Advanced techniques for high-resolution geopotential field determination)

WILLI FREEDEN*, VOLKER MICHEL and HELGA NUTZ
University of Kaiserslautern, Laboratory of Technomathematics, Geomathematics Group, 67653 Kaiserslautern, Germany
(E-mail: freeden@mathematik.uni-kl.de; michel@mathematik.uni-kl.de; nutz@mathematik.uni-kl.de)

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Abstract. The purpose of satellite-to-satellite tracking (SST) and/or satellite gravity gradiometry (SGG) is to determine the gravitational field on and outside the Earth's surface from given gradients of the gravitational potential and/or the gravitational field at satellite altitude. In this paper both satellite techniques are analysed and characterized from a mathematical point of view. Uniqueness results are formulated. The justification is given for approximating the external gravitational field by finite linear combination of certain gradient fields (for example, gradient fields of single-poles or multi-poles) consistent to a given set of SGG and/or SST data. A strategy of modelling the gravitational field from satellite data within a multiscale concept is described; illustrations based on the EGM96 model are given.

Key words: Earth's external gravitational field, multiscale modelling, SST and SGG, uniqueness, well-posedness

1. Introduction

Over the years geoscientists have realized the great complexity of the Earth and its environment. In particular, the knowledge of the gravity potential and its level (equipotential) surfaces have become an important issue. Following the basic principles, various positioning and gravity-field-determination techniques have been designed by geoengineers. Considering the spatial location of the data, one may differentiate between terrestrial (surface), airborne, and spaceborne methods.

The conventional way that is known to the mathematical community is to determine the Earth's gravitational potential using (the magnitudes of) the gravity gradients as boundary values on the Earth's surface (see [1–6]. This approach leads to an exterior oblique boundary-value problem, since the actual surface of the Earth does not coincide with the equipotential surface of the geoid. Provided that both the boundary and the boundary-values are of sufficient smoothness, the oblique boundary-value problem can be solved by well-known (Fredholm) integral-equation methods using the potential of a single layer. These results are summed up in the books by Bitzadse [7] and Miranda [8] (see also [9], [10]).

Observed data material as required for the oblique boundary-value problem from terrestrial source, however, is simply not available globally and will not be for the forseeable future (see Figure 1). Consequently, the determination of the Earth's gravitational potential via an exterior

*Correspondence to W. Freeden

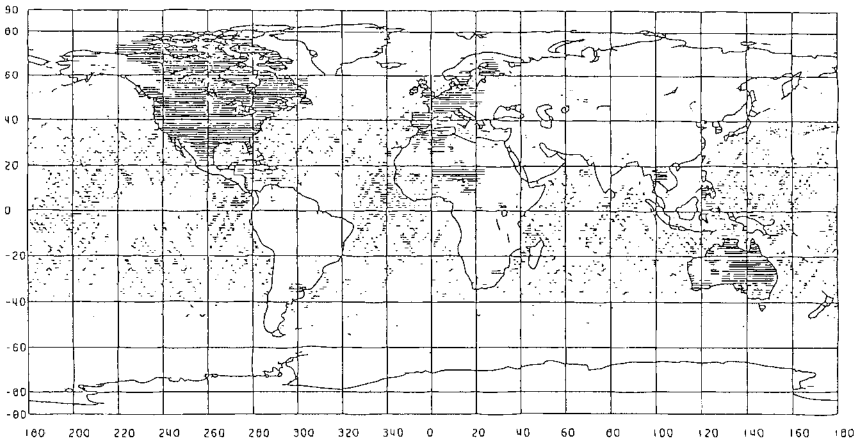

Figure 1. Location of $1^\circ \times 1^\circ$ gravity information of sufficient accuracy (from IAPG, Munich)

oblique boundary-value problem is only of theoretical significance. In practice, a variety of combined techniques must be used to obtain a gravitational potential model of global scale.

Regarding the data types that are observable today we may differentiate between various measurement principles of the gravity field involving derivatives up to the order two, namely *gravity measurements, astronomical positioning, satellite laser ranging, satellite radar altimetry, satellite-to-satellite tracking, and satellite gravity gradiometry*.

Presently available data sources are as follows (*cf.* [11–13]:
(1) *Mean-gravity anomalies*, taken typically over areas of 100×100 km^2 or 50×50 km^2, are derived from terrestrial gravimetry in combination with height measurements and from shipborne gravimetry. Mean values of highly acceptable accuracy are available only for North America, Western Europe, and Australia.
(2) In ocean areas, *satellite radar altimetry* may in some sense be regarded as a direct geoid measuring technique. However, after removing time-varying effects, such as tides, by averaging repeated measurements, the resulting stationary sea surface still deviates from the geoid due to the dynamic ocean topography.
(3) For more than three decades now, several institutions have determined geopotential models from *satellite orbit analysis*. These are derived from the combined analysis of orbits of a large number of mostly non-geophysical satellites with different orbit elements. A variety of tracking techniques can be exploited by laser and Doppler measurements. These models are presented as sets of Fourier (orthogonal) coefficients of a spherical harmonic expansion of the field, and they provide information on the long-wavelength part of the spectrum of the gravitational field only.

There exist combined models of these three data sources, where the best model seems to be the NASA, GSFC, and NIMA Earth Geopotential Model EGM96 (*cf.* [14]). However, neither the above three data sources nor their combination can meet the requirements from physical geodesy, solid-Earth physics, oceanography, geoexploration and -prospection. The traditional techniques of Earth's gravitational-field determination have reached their intrinsic limits. There are essentially two reasons for this fact: An orbit is rather insensitive to local features of the gravitational field, and this insensitivity increases with increasing orbit altitude, and the satellites which can be and are being used are flying at altitudes which are too high for the determination of short wavelengths phenomena. In geophysical reality we have to accept the following principles: the gravitational field of the Earth partially reflects its internal density distribution (*cf.*, for example, [15]). Internal density signatures are mapped to gravitational field signatures. Gravitational signatures smooth out rapidly (*i.e.* exponentially) with increasing distance from the attracting body. As a geoengineering consequence, positioning systems are ideally located as far as possible from Earth, while gravity-field sensors are ideally located as close as possible to Earth. In future, therefore, any advances must rely on space techniques of high-flying positioning systems and low Earth orbiters, because only they provide useful global, regular and dense data sets of high and homogeneous quality.

Fortunately, high spatial resolution can be expected from three actual gravity missions, *viz.* CHAMP (*i.e.*: a German GFZ mission with launch 2000 and an initial altitude of 450 km), GRACE (*i.e.*: a GFZ/NASA advanced mission with launch 2002 and an initial altitude of about 450 km), GOCE (*i.e.*: an ESA high-resolution gravity field mission with planned launch 2005 and an altitude of about 250 km). The observational techniques to be realized, respectively, are satellite-to-satellite tracking in the high-low mode (SST hi-lo), satellite-to-satellite tracking in the low-low mode (SST lo-lo), and satellite gravity gradiometry (SGG).

The scientific justification, the research objectives, and the observational requirements for the gravitational satellite missions CHAMP, GRACE, GOCE have been presented many times by physical geodesists over the past few years, and especially recently in three ESA-reports [11–13]. The basic observable in all three cases is the gravitational acceleration. In the case of SST hi-lo, with the motion of the high-orbiting GPS satellites assumed to be perfectly known, this corresponds to an in situ 3-D acceleration measurement in the low Earth's orbiter (LEO). For SST lo-lo it is the measurement of the acceleration difference over the intersatellite distance and in the line-of-sight (LOS) of the two low Earth's orbiters. In the case of gradiometry it is the measurement of acceleration differences in 3-D over the time baseline of the gradiometer.

In short, we have the following characterization of the observational variants:

SST hi-lo:	3-D acceleration = gravitational gradient
SST lo-lo:	acceleration difference = difference in gradient
SGG:	differential = gradient of gradient

In a mathematical sense it is a transition from the first derivatives of the gravitational potential via a difference in first derivatives to the second derivatives. The guiding parameter that determines the sensitivity with respect to the spatial scales of the Earth's gravitational potential is the distance between the test masses, being almost infinite for SST hi-lo and almost zero for SGG. The purpose of these three measurement concepts are to counteract the natural attenuation of the gravitational field with altitude by differential measurement, where the gravitational sensitivity increases with decreasing distance.

In what follows satellite-to-satellite tracking (SST) and satellite gravity gradiometry (SGG) are characterized from a mathematical point of view. Uniqueness results are formulated. Moreover, the mathematical justification is given for approximating the external gravitational field by finite linear combinations of certain gradient fields (for example, gradient fields of single poles, multipoles, and kernel functions) by use of a prescribed set of SST and/or SGG data.

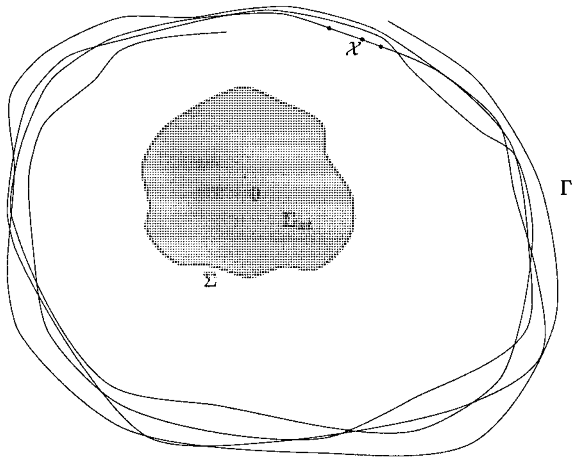

Figure 2. Illustration of the sets Σ and Γ

2. Formulation of the problems

We begin by introducing the mission concepts of CHAMP, GRACE, and GOCE in more detail.

2.1. THE SST PROBLEMS

The purpose of *high-low satellite-to-satellite tracking* (hi-lo SST) by use of the Global Positioning System (GPS) (as realized *e.g.* by the recently (2000) launched German satellite CHAMP (= Challenging Mini-Satellite Payload for Geophysical Research and Application) of the GeoForschungsZentrum (GFZ) Potsdam) is to develop the geopotential field from measured ranges (geometrical distances) between a low Earth orbiter (LEO) and the high-flying GPS-satellites. Next, hi-lo SST is discussed from a mathematical point of view as the problem of determining the external gravitational field of the Earth from a given set of gradient vectors at the altitude of the low Earth orbiter (LEO).

In order to translate hi-lo SST into a mathematical formulation (see [16, pp. 259–271], [17], for alternative approaches [11–13, 18–27]) we start from the following geometrical situation (*cf.* Figure 2): Let the surface Σ of the Earth $\overline{\Sigma_{\mathrm{int}}}$ and the orbital set Γ of the low Earth orbiter (LEO) be given in such a way that Γ is a strict subset of the Earth's exterior Σ_{ext} satisfying

$$\sigma^{\sup} = \sup_{x\in\Sigma} |x| < \gamma = \inf_{x\in\Gamma} |x|. \tag{1}$$

The arrangement of the GPS-satellites is such that at least four satellites are simultaneously visible above the horizon anywhere on the Earth's surface Σ and the orbit Γ of the low Earth orbiter as well, all the time. Moreover, the GPS-satellites are supposed to be placed in six circular orbits Ω_{γ_i} of radii γ_i, $i = 1, \ldots, 6$, around the origin with $\gamma_i \gg \gamma$, $i = 1, \ldots, 6$; and n be the total number of GPS-satellites. To every LEO-position $x \in \Gamma$, therefore, there exist at least $m(\geq 4)$ visible GPS-satellites located at $y_{l_1}, \ldots, y_{l_m}$, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$, such that the geometrical distances (ranges) $d_{l_i} = |x - y_{l_i}|$, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$, are measurable. Since the orbits of the GPS-satellites are assumed to be known, the coordinates of the low Earth orbiter (LEO) located at $x \in \Gamma$ can be derived from simultaneous

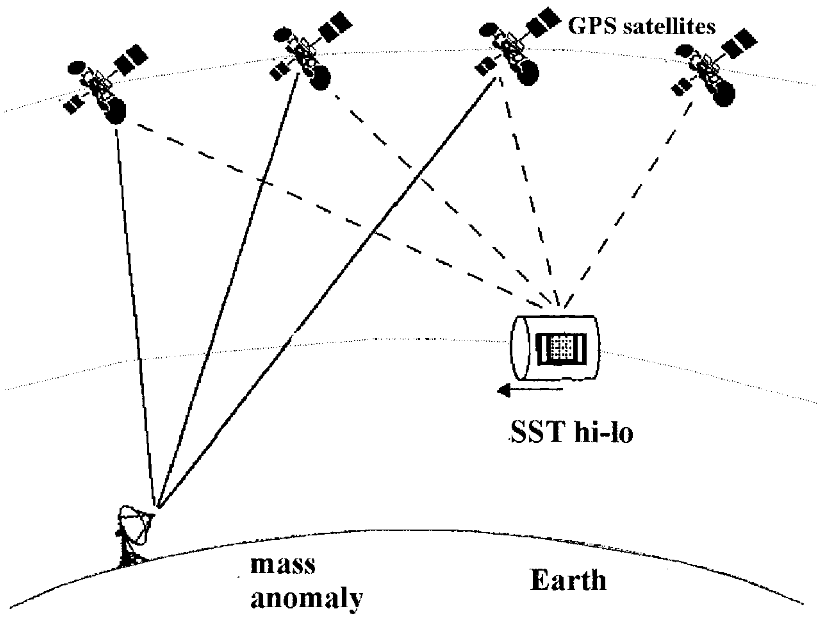

Figure 3. Satellite-to-satellite tracking in the high-low mode: the CHAMP concept (from [12, p. 24])

range measurements to the satellites. From this the relative positions of the satellites at x and y_{l_i}, *i.e.*

$$p_{l_i} = x - y_{l_i}, \quad l_i \in \{1, \ldots, n\}, \quad i = 1, \ldots, m, \tag{2}$$

become available at time t. The relative velocities v_{l_i} and accelerations a_{l_i} are obtainable by differentiating the relative positions with respect to t. We may assume that the measurements are produced at a sufficiently dense rate so that (numerical) differentiation can be performed without any difficulty. The interesting expressions now are the relative accelerations a_{l_i}, $i = 1, \ldots, m$, all of which are determined for inertial motion (in accordance with the Newton-Euler equation) by the gravitational field only and may be equated by the difference of the gradient field of the geopotential, V, here evaluated at the locations of x and y_{l_i}, $l_i \in \{1, \ldots, n\}$ for $i = 1, \ldots, m$. To be more specific,

$$a_{l_i}(x) = (\nabla V)(x) - (\nabla V)(y_{l_i}), \quad x \in \Gamma, \tag{3}$$

$i = 1, \ldots, m$. (Note that the gravitational force is considered now to be independent of time t at a certain position. In other words, we assume here that the time-like variations of the field are so slow as to be negligible.) From (3) it follows that

$$(\nabla V)(x) = \sum_{i=1}^{m} \alpha_i \left(a_{l_i}(x) + (\nabla V)(y_{l_i}) \right), \quad x \in \Gamma, \tag{4}$$

for all selections $(\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$ satisfying $\sum_{i=1}^m \alpha_i = 1$. The influence of the Global Positioning System (GPS) to the choice of the coefficients $\alpha_1, \ldots, \alpha_m$ will not be investigated here. (Usually, in practice, $(\nabla V)(y_{l_i})$ are supposed to be so small as to be negligible).

Loosely phrased, the mathematical formulation of the hi-lo SST problem now reads as follows:

Let there be known the gradient vectors

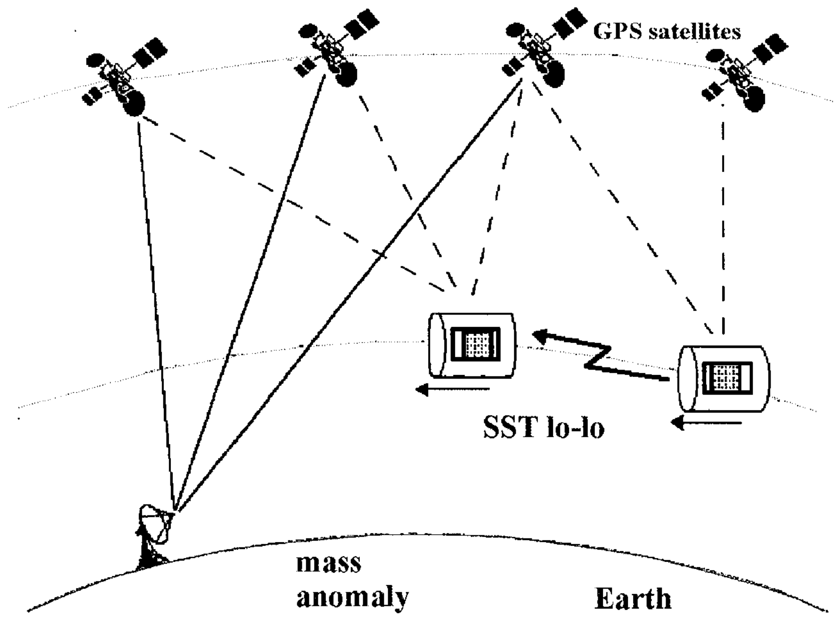

Figure 4. Satellite-to-satellite tracking in the low-low mode: the GRACE concept (from [12] p. 25)

$$v(x) = (\nabla V)(x), \quad x \in \chi, \tag{5}$$

for a subset $\chi \subset \Gamma$ of points at the flight positions of the low Earth orbiter (LEO). Find an approximation u of the geopotential field v on $\overline{\Sigma_{\text{ext}}}$, i.e. on and outside the Earth's surface, such that the geopotential field v and its approximation u are in ε-accuracy on $\overline{\Sigma_{\text{ext}}}$ (with respect to the uniform topology in $\overline{\Sigma_{\text{ext}}}$) so that $v(x) = u(x)$ for all $x \in \chi$.

The problem of knowing the vectors $(\nabla V)(y_{l_i})$, $i = 1, \ldots, m$, in Equation (3) is no longer relevant, if *low-low satellite-to-satellite tracking* (briefly, lo-lo SST) will be used (as planned by the future GFZ/NASA 'two satellite configuration' GRACE (= Gravity Recovery and Climate Experiment) (2001)). By the tandem-mode procedure of lo-lo SST (see the explanations in [11–13]) the vectors a_{l_i}, $i = 1, \ldots, m$, are measurable at two different positions x and x^* with $x^* = x + h(x)$, $x \in \Gamma$, where $h : \Gamma \to \mathbb{R}^3$ is the difference vector field between the two satellite positions (*i.e.* $|h(x)| \geq \iota > 0$ with ι denoting the intersatellite range). Consequently, the mathematical scenario of the lo-lo SST problem is characterized as follows:

Let there be known the vectors $v(x) = (\nabla V)(x)$ and $\tilde{v}(x) = v(x + h(x)) = (\nabla V)(x + h(x))$, $x \in \chi$, for a subset $\chi \subset \Gamma$. Find an approximation u of v on $\overline{\Sigma_{\text{ext}}}$, such that v and u are in ε-accuracy (with respect to the uniform topology in $\overline{\Sigma_{\text{ext}}}$), so that $v(x) - v(x + h(x)) = u(x) - u(x + h(x))$ for all $x \in \chi$.

2.2. The SGG problem

As we already mentioned, the current knowledge of the Earth's gravity field, as derived from various observing techniques, is incomplete. Within reasonable time, substantial improvement can only come from exploiting new approaches based on satellite-gravity-observation methods. The purpose now is to provide an overview of the satellite technique SGG to be realized in the ESA satellite GOCE (= Gravity Field and Steady-State Ocean Circulation Mission)

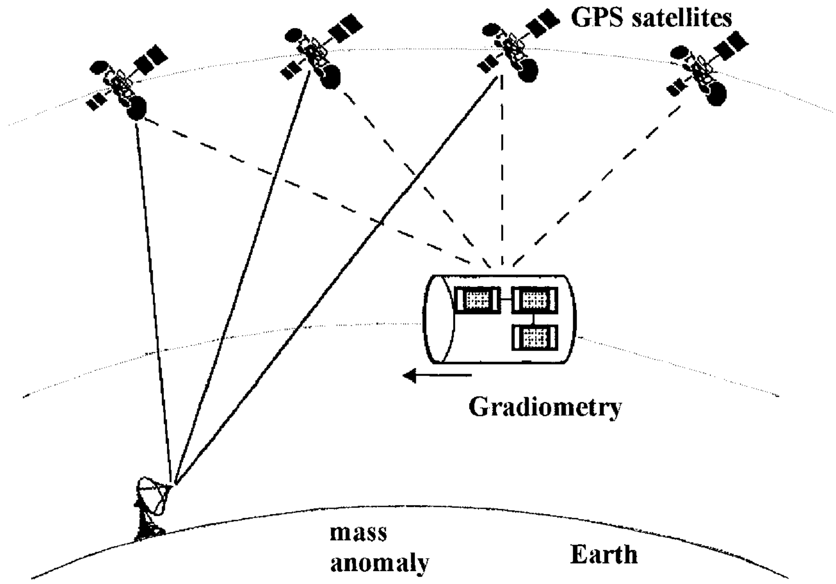

Figure 5. Satellite Gradiometry: the GOCE concept (from [12] p. 26)

that has a planned launch in the year 2005. The concept considered for the GOCE mission (*cf.* [13]) is satellite gravity gradiometry (SGG), *i.e.* the measurement of the relative acceleration of test masses at different locations inside one satellite.

In an idealized situation, free from non-gravitational influences, the acceleration vector of a proof-mass in free fall at the centre x of mass of a space vehicle is, according to Newton's law, equal to the gradient of the gravitational potential: $v = \nabla V$ (for further details see [22, 23, 26–31]). Considering now the motion of a second proof-mass at y close to x relative to the first one, its acceleration is, in a linearised sense, given by $v(y) \simeq v(x) + \mathbf{v}(x)(y - x)$. The matrix $\mathbf{v}(x) = (\nabla v)(x)$ is the Hesse matrix $\mathbf{v}(x) = \nabla^{(2)} V(x) = (\nabla \otimes \nabla) V(x)$ consisting of all second-order derivatives of the Earth's gravitational potential V. Because of its tensor properties, $\mathbf{v}$ is called the gravitational tensor. In other words, measurements of the relative accelerations between two test masses provide information about the second-order partial derivatives of the gravitational potential V. In an ideal observational situation, the full Hesse matrix is available by an array of test masses.

An illustrative view of satellite gradiometry based on Newton's theory of gravitation is as follows (*cf.* [32, p. 116]: History has it that Newton, when working on his law of gravitation, was inspired by a falling apple. Referring to the theory of gravitation in as the tale of the falling apple, it would be appropriate for us to view gradiometry as the story of two falling apples. In their famous book, C.W. MISNER *et al.* [33, pp. 195–218]) made this point clear. In one of their examples it is shown, that measuring the relative distance between the shortest paths taken by two ants walking at the skin of an apple, from two adjacent begin- to two adjacent end-points, the geometry of its curved surface can be derived. Translated to our case, shortest path means geodetic or free fall of two test particles (apples), from the relative motion of which the geometry of the curved space can be inferred, curved by the gravitational field of the Earth: if gravity is interpreted in terms of geometry in the sense of Einstein, gradiometry, when all nine observable gradient components are measured in a point, shows the complete local geometry of the relative motion of adjacent proof-masses in free fall. However, it is more

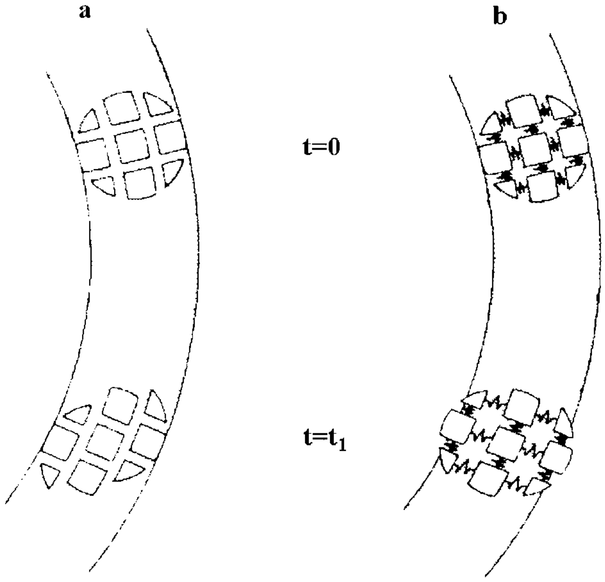

Figure 6. Proof masses in orbit: (a) independent and (b) constrained by springs (from [32])

practical to constrain their relative motion by highly sensitive springs and measure instead the tension and compression of the springs. This is equivalent to saying that a gradiometer is realized by a coupled system of highly sensitive micro-accelerometers. (A gradiometer of this kind is envisaged for the already mentioned GOCE mission planned by ESA (*cf.* [13]) to produce a coverage of the entire Earth with measurements).

In conclusion, the mathematical formulation of the SGG-problem (after separating all non-gravitational influences) reads as follows:

Let there be known from the gravitational field v of the Earth the gradients

$$\mathbf{v}(x) = (\nabla v)(x), \quad x \in \chi, \tag{6}$$

for a subset χ of the orbit Γ of the low Earth orbiter (LEO). Find an approximation u on $\overline{\Sigma_{\text{ext}}} = \Sigma \cup \Sigma_{\text{ext}}$, i.e. on the Earth's surface and in the outer space Σ_{ext}, such that v and its approximation u are in ε-accuracy on $\overline{\Sigma_{\text{ext}}}$ (with respect to the uniform topology) so that

$$(\nabla v)(x) = \mathbf{v}(x) = (\nabla u)(x)$$

for all $x \in \chi$.

3. Notational background

Let us begin by introducing some notations that will be used throughout this paper. We consider $\mathbb{R}^3$ to be equipped with the canonical inner product $\cdot$ and the associated norm $|\cdot|$. Using $\varepsilon^1, \varepsilon^2, \varepsilon^3$ as canonical orthonormal basis in $\mathbb{R}^3$ we may represent each element $x \in \mathbb{R}^3$ in Cartesian coordinates as follows: $x = \Sigma_{i=1}^3 (x \cdot \varepsilon^i)\varepsilon^i$.

If G is a set of points in $\mathbb{R}^3$, ∂G will denote its *boundary*. The set $\overline{G} = G \cup \partial G$ will be called the *closure* of G. A set $G \subset \mathbb{R}^3$ will be called a *region* if it is open and connected.

The restriction of a function f to a subset M of its domain is denoted by $f|M$; for a set L of functions we set $L|M = \{f|M \big| f \in L\}$.

A function f possessing k continuous derivatives on the whole domain is said to be of class $C^{(k)}$ (note that $C^{(k)}$, $c^{(k)}$, $\mathbf{c}^{(k)}$ is used for scalar-valued, vector-valued, and tensor-valued functions, respectively).

A surface Σ is called *regular*, if it satisfies the following properties:

(i) Σ divides the three-dimensional Euclidean space $\mathbb{R}^3$ into the bounded region Σ_{int} (*inner space*) and the unbounded region Σ_{ext} (*outer space*) defined by $\Sigma_{\text{ext}} = \mathbb{R}^3 - \overline{\Sigma_{\text{int}}}$.

(ii) Σ is a closed and compact surface with no double points.

(iii) The origin 0 is contained in Σ_{int}.

(vi) Σ is a $C^{(2)}$-surface, *i.e.* Σ is locally $C^{(2)}$-smooth.

From this definition it is clear that all (geophysically relevant) Earth models are included. Regular surfaces are, for example, a sphere, an ellipsoid, a geoid, and the (sufficiently smooth) real Earth's surface.

$\text{Pot}(\Sigma_{\text{ext}})$ denotes the space of functions $V : \Sigma_{\text{ext}} \to \mathbb{R}$ with the following properties:

(i) V is twice continuously differentiable in Σ_{ext}: $V \in C^{(2)}(\Sigma_{\text{ext}})$,

(ii) V satisfies Laplace's equation in Σ_{ext}: $\Delta V = 0 \text{in} \Sigma_{\text{ext}}$,

(iii) V is regular at infinity:

$$|V(x)| = O\left(\frac{1}{|x|}\right), \quad |\nabla V(x)| = O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty.$$

We denote by $\text{Pot}^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right)$ the space of all functions $V : \overline{\Sigma_{\text{ext}}} \to \mathbb{R}$ such that V is a member of class $C^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right)$ and $V\big|\Sigma_{\text{ext}}$ satisfies, in addition, the properties (i), (ii), (iii) of a function of class $\text{Pot}(\Sigma_{\text{ext}})$. Briefly formulated,

$$\text{Pot}^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right) = \text{Pot}(\Sigma_{\text{ext}}) \cap C^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right).$$

By $\text{pot}(\Sigma_{\text{ext}})$ we denote the space of vector fields $v : \Sigma_{\text{ext}} \to \mathbb{R}^3$ satisfying the following properties:

(i) v is continuously differentiable in Σ_{ext}: $v \in c^{(1)}(\Sigma_{\text{ext}})$,

(ii) v is a harmonic vector field (*cf.* [34] p. 24) in Σ_{ext}:
$\text{div}\, v = 0, \quad \text{curl}\, v = 0 \quad \text{in} \Sigma_{\text{ext}}$,

(iii) v is regular at infinity:

$$|v(x)| = o(1), \quad |x| \to \infty.$$

In analogy to the scalar notation we let

$$\text{pot}^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right) = \text{pot}(\Sigma_{\text{ext}}) \cap c^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right).$$

By $\mathbf{pot}(\Sigma_{\text{ext}})$ we denote the space of tensor fields $\mathbf{v} : \Sigma_{\text{ext}} \to \mathbb{R}^{3\times3}$ satisfying the following properties:

(i) $\mathbf{v}$ is continuously differentiable in Σ_{ext}: $\mathbf{v} \in \mathbf{c}^{(1)}(\Sigma_{\text{ext}})$,

(ii) $\mathbf{v}$ is a harmonic tensor field in Σ_{ext}:
$$\text{div}\,\mathbf{v} = 0, \quad \text{curl}\,\mathbf{v} = 0 \quad \text{in}\,\Sigma_{\text{ext}},$$

(iii) $\mathbf{v}$ is regular at infinity:
$$|\mathbf{v}(x)| = o(1), \quad |x| \to \infty.$$

In analogy to the scalar and vectorial approach we let

$$\mathbf{pot}^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right) = \mathbf{pot}(\Sigma_{\text{ext}}) \cap \mathbf{c}^{(k)}\left(\overline{\Sigma_{\text{ext}}}\right).$$

As it is well-known, every member $\mathbf{v} \in \mathbf{pot}(\Sigma_{\text{ext}})$ can be represented as a gradient field $\mathbf{v} = \nabla v$, where v is of class $\text{pot}(\Sigma_{\text{ext}})$, and *vice versa* (see, for example, [19, 34, 35]). As a consequence of this, in connection with the fact that every $v \in \text{pot}\,(\Sigma_{\text{ext}})$ can be represented as a gradient field $v = \nabla V$ with $V \in \text{Pot}\,(\Sigma_{\text{ext}})$, we finally get that a tensor field $\mathbf{v} \in \mathbf{pot}\,(\Sigma_{\text{ext}})$ can be represented as the Hesse tensor of a scalar field $V \in \text{Pot}\,(\Sigma_{\text{ext}})$:

$$\mathbf{v} = \nabla^{(2)} V = (\nabla \otimes \nabla)\, V.$$

Obviously, $\mathbf{v} \in \mathbf{pot}\,(\Sigma_{\text{ext}})$ of the form $\mathbf{v} = \sum_{i,k=1}^{3} V_{ik}\varepsilon^i \otimes \varepsilon^k$ fulfills $V_{ik} \in \text{Pot}\,(\Sigma_{\text{ext}})$.

$C^{(0)}(\Sigma)$ is the Banach space with the norm defined by

$$||F||_{C^{(0)}(\Sigma)} = \sup_{x\in\Sigma} |F(x)|.$$

In $C^{(0)}(\Sigma)$ we are able to introduce the (L^2-)inner product

$$(F, G)_{\mathrm{L}^2(\Sigma)} = \int_{\Sigma} F(x)G(x)\mathrm{d}\omega(x),$$

where $\mathrm{d}\omega(x)$ (or, when confusion is not likely to arise, $\mathrm{d}\omega$) denotes the surface element. The inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Sigma)}$ implies the norm

$$\|F\|_{\mathrm{L}^2(\Sigma)} = \sqrt{(F, F)_{\mathrm{L}^2(\Sigma)}}.$$

The space $(C^{(0)}(\Sigma), (\cdot,\cdot)_{\mathrm{L}^2(\Sigma)})$ is a pre-Hilbert space. For every function $F \in C^{(0)}(\Sigma)$ we have the norm-estimate

$$\|F\|_{\mathrm{L}^2(\Sigma)} \leq C\|F\|_{C^{(0)}(\Sigma)}, \quad C = \sqrt{\|\Sigma\|}. \tag{7}$$

By $\mathrm{L}^2(\Sigma)$ we denote the space of (Lebesgue) square-integrable functions on the boundary Σ. $\mathrm{L}^2(\Sigma)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Sigma)}$ and a Banach space with respect to the norm $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$, $\mathrm{L}^2(\Sigma)$ is the completion of $C^{(0)}(\Sigma)$ with respect to the norm $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$.

For later use we finally introduce the concept of fundamental systems:

DEFINITION 3.1 *A system* $\mathcal{Y} = (y_n)_{n=0,1,\ldots} \subset \Sigma_{\text{int}}$ *is called a fundamental system in* Σ_{int}, *if* $F : \Sigma_{\text{int}} \to \mathbb{R}$ *with* $F \in C^{(2)}(\Sigma_{\text{int}})$, $\Delta F = 0$ *in* Σ_{int}, *and* $F(y_n) = 0$ *for* $n = 0, 1, \ldots$ *implies* $F = 0$ *in* Σ_{int}. *Analogously, a system* $\mathcal{Y} = (y_n)_{n=0,1,\ldots} \subset \Sigma_{\text{ext}}$ *is called a fundamental system* *in* Σ_{ext}*, if* $F : \Sigma_{\text{ext}} \to \mathbb{R}$ *with* $F \in C^{(2)}(\Sigma_{\text{ext}})$*,* $\Delta F = 0$ *in* Σ_{ext}*,* F *is regular at infinity, and* $F(y_n) = 0$ *for* $n = 0, 1, \ldots$ *implies* $F = 0$ *in* Σ_{ext}.

4. Uniqueness of the satellite problems

Our considerations start with the study of uniqueness corresponding to an *infinite* system $\chi \subset \Gamma$ of known satellite data.

4.1. Uniqueness of the SST problem

First, we are concerned with the following theorem which provides the uniqueness of the SST problem from given vector values (*cf.* [36]).

THEOREM 4.1 Let Σ (*i.e. the Earth's surface) be regular. Suppose that* χ *(i.e. the subset of observational points on the satellite orbit* Γ*) is a fundamental system in* Σ_{ext} *with*

$$\sigma^{\text{sup}} = \sup_{x \in \Sigma} |x| < \gamma \leq \inf_{x \in \chi} |x|. \tag{8}$$

If v *is of class* $\text{pot}^{(0)}\left(\overline{\Sigma_{\text{ext}}}\right)$ *such that* $v(x) = 0$*,* $x \in \chi$*, then* $v = 0$ *in* $\overline{\Sigma_{\text{ext}}}$.

Proof. Any field $v \in \text{pot}^{(0)}(\overline{\Sigma_{\text{ext}}})$ can be expressed in the form ∇V, $V \in \text{Pot}^{(1)}(\overline{\Sigma_{\text{ext}}})$, hence, the coordinate functions $v \cdot \varepsilon^i$, $i = 1, 2, 3$, satisfy

$$\Delta(v \cdot \varepsilon^i) = \Delta(\varepsilon^i \cdot \nabla)V = (\varepsilon^i \cdot \nabla)\Delta V = 0 \tag{9}$$

in Σ_{ext}, since the harmonic function V is arbitrarily often differentiable in Σ_{ext}. Moreover, according to our assumption, $(\varepsilon^i \cdot \nabla)V(x) = 0$ for all points x of the fundamental system χ in Σ_{ext}. This implies $v \cdot \varepsilon^i = 0$ in Σ_{ext}, $i = 1, 2, 3$, as required. □

Furthermore, we are able to verify the following result (for a similar theorem see [16]).

THEOREM 4.2 *Suppose that* χ *is a fundamental system in* Σ_{ext} *satisfying (8). If* v *is a field of class pot*$^{(0)}(\overline{\Sigma_{\text{ext}}})$ *with* $(-x) \cdot v(x) = 0$*,* $x \in \chi$*, then* $v = 0$ *in* $\overline{\Sigma_{\text{ext}}}$.

Proof. Again, we base our arguments on the identity $v = \nabla V$. From our assumptions it is clear that there exists a sphere B with radius β around the origin such that $\sigma^{\text{sup}} = \sup_{x \in \Sigma} |x| < \beta < \gamma$, *i.e.* B_{ext} is a strict subset of Σ_{ext}. Outside the sphere B the potential $V \in \text{Pot}^{(\infty)}(\overline{B_{\text{ext}}})$ may be expanded in terms of outer harmonics (see Example 5.1)

$$V(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge}(n, k) H_{n,k}(\beta; x), \quad x \in B_{\text{ext}}, \tag{10}$$

where $V^{\wedge}(n, k)$, $n = 0, 1, \ldots; k = 1, \ldots, 2n + 1$, are the expansion coefficients

$$V^{\wedge}(n, k) = \int_B V(x) H_{n,k}(\beta; x) \mathrm{d}\omega(x). \tag{11}$$

It is not hard to see that

$$-\frac{x}{|x|} \cdot (\nabla V)(x) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{n+1}{|x|} V^{\wedge}(n, k) H_{n,k}(\beta; x), \quad x \in B_{\text{ext}}.$$

Hence, $x \mapsto (-x) \cdot (\nabla V)(x)$, $x \in B_{\mathrm{ext}}$, is a function of class $\mathrm{Pot}^{(\infty)}(\overline{B_{\mathrm{ext}}})$ from which we know the assumption that $(-x) \cdot (\nabla V)(x) = 0$ for all $x \in \chi$. Consequently, we obtain

$$\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} V^{\wedge}(n,k)(n+1) H_{n,k}(\beta; x) = 0, \quad x \in \chi. \tag{12}$$

Since χ is assumed to be a fundamental system in B_{ext}, Equation (12) holds true in B_{ext}. The theory of spherical harmonics then tells us that $V^{\wedge}(n,k)(n+1) = 0$, hence, $V^{\wedge}(n,k) = 0$ for all $n = 0, 1, \ldots; k = 1, \ldots, 2n+1$. This yields $V = 0$ in B_{ext}. By analytical continuation we get $V = 0$ in Σ_{ext}, hence, $v = 0$ in Σ_{ext}. This is the desired result. □

Theorem 4.2 means that the Earth's external gravitational field is uniquely recoverable from first (negative radial) derivatives corresponding to a fundamental system χ of the satellite orbit. In other words, the Earth's external gravitational field is uniquely detectable on and outside the Earth's surface Σ from GPS-SST data corresponding to a system of gradient vectors given on a fundamental system χ on the satellite orbit Γ.

From potential theory it is clear that analogous uniqueness theorems (as mentioned before) cannot be deduced for the '*actual*' hi-lo SST problem of finding the external gravitational field of the Earth from a *finite* subsystem χ on the satellite orbit Γ. In Section 7, however, we shall show that, given the SST data for a finite subset $\chi \subset \Gamma$, we are able to find, for every value $\varepsilon > 0$, an approximation u of the external gravitational field v of the Earth in ε-accuracy so that u additionally is consistent to the SST data on the finite subsystem χ.

4.2. Uniqueness of the SGG problem

Our considerations start with the problem of uniqueness corresponding to an *infinite* system $\chi \subset \Gamma$ of known SGG data.

THEOREM 4.3 *Suppose that χ (i.e. the subset of observational points of the satellite orbit Γ) is a fundamental system in Σ_{ext} such that (8) holds true. If* $\mathbf{v}$ *is of class* $\mathbf{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$ *with* $\mathbf{v}(x) = 0, \quad x \in \chi$, *then the associated field* $v \in \mathrm{pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$ *with* $\mathbf{v} = \nabla v$ *satisfies* $v = 0$ *in* $\overline{\Sigma_{\mathrm{ext}}}$.

Proof. Any field $\mathbf{v}$ of the class $\mathbf{pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ can be expressed in the form $\nabla^{(2)} V = (\nabla \otimes \nabla) V$, $V \in \mathrm{Pot}^{(2)}(\overline{\Sigma_{\mathrm{ext}}})$. Furthermore, the coordinate functions $V_{ij} = {\varepsilon^i}^T \mathbf{v} \varepsilon^j$, $i, j \in \{1, 2, 3\}$, satisfy $\Delta V_{ij} = 0$ in Σ_{ext}. This implies $V_{ij} = 0$ in Σ_{ext}, $i, j \in \{1, 2, 3\}$, because of the definition of a fundamental system. From $\mathbf{v} = (\nabla \otimes \nabla) V = 0$ we finally get $V = 0$ in Σ_{ext} and, thus, $v = \nabla V = 0$, as required. □

In other words, the Earth's external gravitational field v is uniquely detectable on and outside the Earth's surface Σ if SGG data (*i.e.* second-order derivatives of the Earth's gravitational potential V) are given on a fundamental system χ (on the satellite orbit).

Furthermore, we are able to verify the following result:

THEOREM 4.4 *Suppose that χ is a fundamental system in Σ_{ext} satisfying (8). If* $\mathbf{v}$ *is a field of class* $\mathbf{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$ *with* $x \cdot (\mathbf{v}(x) x) = 0$, $x \in \chi$, *then* $v = 0$ *in* $\overline{\Sigma_{\mathrm{ext}}}$, *where* $\mathbf{v} = \nabla v$.

Proof. Clearly, we base our arguments on the identity $\mathbf{v} = (\nabla \otimes \nabla) V$. From our assumptions it is clear that there exists a sphere B with radius β around the origin such that

$\sigma^{\sup} = \sup_{x\in\Sigma}|x| < \underline{\beta} < \gamma$, *i.e.* B_{ext} is a strict subset of Σ_{ext}. Outside the sphere B the potential $V \in \text{Pot}^{(\infty)}(\overline{B_{\text{ext}}})$ may be expanded in terms of outer harmonics

$$V(x) = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} V^{\wedge}(n,k)H_{n,k}(\beta;x), \quad x \in B_{\text{ext}}, \tag{13}$$

where $V^{\wedge}(n,k)$ are the orthogonal coefficients. By elementary calculations we get

$$\begin{aligned}\frac{x}{|x|}\cdot\left(\nabla^{(2)}V(x)\frac{x}{|x|}\right) &= \frac{x}{|x|}\cdot\left((\nabla\otimes\nabla V)(x)\frac{x}{|x|}\right)\\ &= \left(\frac{x}{|x|}\cdot\nabla_x\right)\left(\frac{x}{|x|}\cdot\nabla_x\right)V(x)\\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}\frac{(n+1)(n+2)}{|x|^2}V^{\wedge}(n,k)H_{n,k}(\beta;x),\end{aligned}$$

$x \in B_{\text{ext}}$. Hence $x \mapsto x\cdot((\nabla\otimes\nabla V)(x)x)$, $x \in B_{\text{ext}}$, is a harmonic function in B_{ext}. In accordance with our assumption $x\cdot((\nabla\otimes\nabla V)(x)x) = 0$, $x \in \chi$, we thus obtain

$$\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} V^{\wedge}(n,k)(n+1)(n+2)H_{n,k}(\beta;x) = 0, \quad x \in \chi. \tag{14}$$

Since χ is a fundamental system in B_{ext}, Equation (14) holds true in B_{ext}. The theory of spherical harmonics then tells us that $V^{\wedge}(n,k)(n+1)(n+2) = 0$, hence, $V^{\wedge}(n,k) = 0$ for $n = 0, 1, \ldots; k = 1, \ldots, 2n+1$. This yields $V = 0$ in B_{ext}. By analytical continuation we have $V = 0$ in Σ_{ext}, and, hence, $v = \nabla V = 0$ in Σ_{ext}.
□

Theorem 4.4 means that the Earth's external gravitational field is uniquely recoverable from 'second radial derivatives' corresponding to a fundamental system $\chi \subset \Gamma$.

From potential theory it is again clear that analogous uniqueness theorems (as mentioned before) cannot be deduced for the '*actual*' SGG problem of finding the external gravitational field v of the Earth from a *finite* subsystem χ on the satellite orbit Γ. In what follows, however, we shall show that, given the SGG data for a finite subset $\chi \subset \Gamma$, we are able to find, for every value $\varepsilon > 0$, an approximation u of the external gravitational field v of the Earth in ε-accuracy so that u additionally is consistent to the SGG data on the finite subsystem χ.

5. Scalar approximation

Let A be a sphere inside (the Earth) Σ_{int} of radius α centered at the origin 0 (*cf.* Figure 7) with

$$\alpha < \sigma^{\inf} = \inf_{x\in\Sigma}|x|.$$

We consider simultaneously the outer space A_{ext} of the sphere A and the outer space Σ_{ext}. Of course, $\overline{\Sigma_{\text{ext}}} \subset A_{\text{ext}}$.

A system (Φ_n), $\Phi_n \in \mathrm{L}^2(A)$, $n = 0, 1, \ldots$, is called *complete* in the Hilbert space $\mathrm{L}^2(A)$, if it satisfies the following property: For every $\Phi \in \mathrm{L}^2(A)$, the condition

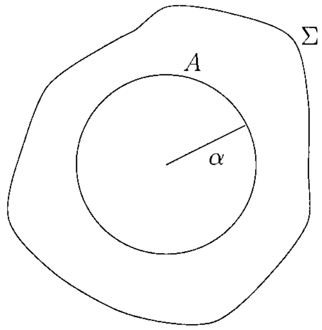

Figure 7. Illustration of the sets A and Σ

$$(\Phi, \Phi_n)_{\mathrm{L}^2(A)} = \int_A \Phi(x)\Phi_n(x)\mathrm{d}\omega(x) = 0$$

for all $n = 0, 1, \ldots$ implies $\Phi = 0$ (in the sense of $\mathrm{L}^2(A)$).

In scalar potential theory a large number of systems $(\tilde{\Phi}_n)_{n=0,1,\ldots}$ is known satisfying $\tilde{\Phi}_n \in \mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$, $\tilde{\Phi}_n|A = \Phi_n$, $n = 0, 1, \ldots,$ and $(\Phi_n)_{n=0,1,\ldots}$ is complete in $\mathrm{L}^2(A)$ (see, for example, [9, 10, 37–39]).

The most important system in the geosciences is the system of outer harmonics (*i.e.* multipoles).

Example 5.1. *Let* $(H_{n,k}(\alpha;\cdot))_{\substack{n=0,1,\ldots\\ k=1,\ldots,2n+1}}$ *be the system of outer harmonics given by*

$$H_{n,k}(\alpha; x) = \frac{1}{\alpha}\left(\frac{\alpha}{|x|}\right)^{n+1} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \overline{A_{\mathrm{ext}}},$$

where $\{Y_{n,k}\}_{\substack{n=0,1,\ldots\\ k=1,\ldots,2n+1}}$ *is a (maximal) system of spherical harmonics being orthonormal with respect to the* L^2*-inner product over the unit sphere. Then*

$$\left(H_{n,k}(\alpha; x)\Big|_{x\in A}\right)_{n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(A)$.

In order to illustrate the role of single poles we use the concept of fundamental systems.

Example 5.2. *Suppose that* $\mathcal{Y} = (y_n)_{n=0,1,\ldots}$ *is a fundamental system in* A_{int}. *Denote by*

$$M(x, y_n) = \frac{1}{|x - y_n|}, \quad x \in \overline{A_{\mathrm{ext}}},$$

the single-poles (mass–points) at $y_n \in \mathcal{Y}$, $n = 0, 1, \ldots$. *Then*

$$\left(M(x, y_n)\Big|_{x\in A}\right)_{n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(A)$.

It should be mentioned that the completeness of outer harmonics in $\mathrm{L}^2(A)$ is a well-known fact in potential theory (see, for example, [35, 40–42]). For mass-point systems the completeness property has been proved already in [37] (in fact, the completeness can be verified, even for arbitrary fundamental systems $(y_n)_{n=0,1,\ldots}$ in A_{int} and inner spaces of regular surfaces Σ).

Some examples of fundamental systems in A_{int} should be listed below:

(i) If $\mathcal{Y}$ is a countable dense set of points on a regular surface $\Xi \subset A_{\mathrm{int}}$ with $\Xi_{\mathrm{int}} \subset A_{\mathrm{int}}$, then $\mathcal{Y}$ is a fundamental system in A_{int}.

(ii) If $\mathcal{Y}$ is a countable dense set of points in a region $\Xi_{\mathrm{int}} \subset A_{\mathrm{int}}$, with Ξ being a regular surface satisfying $\mathrm{dist}(\Xi, A) > 0$, then $\mathcal{Y}$ is a fundamental system in A_{int}.

Remark 5.3. Consider the fundamental system $\mathcal{Y} = (y_n)_{n=0,1,\ldots}$ in A_{int} generated by $\overline{\mathcal{Y}} = \left(\overline{y}_n\right)_{n=0,1,\ldots}$ as follows:

(i) $\left(\overline{y}_n\right)$ is a countable dense system on the (real Earth's) surface $\Sigma \subset A_{\mathrm{ext}}$,

(ii) $(y_n)_{n=0,1,\ldots}$ is obtained by letting $y_n = \frac{\alpha^2}{|\overline{y}_n|^2}\overline{y}_n$.

This set seems to be a suitable point system for practical purposes (cf. the numerical experiences in [38]).

Further complete systems can be obtained by using $(K(x, y_n))_{n=0,1,\ldots}$ with

$$K(x,y) = \frac{1}{|x|}\sum_{k=0}^{\infty}\frac{2k+1}{4\pi\alpha^2}\sigma_k\left(\frac{|y|}{|x|}\right)^k P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right),\; x \in \overline{A_{\mathrm{ext}}},\; y \in \mathcal{Y} \subset A_{\mathrm{int}}, \tag{15}$$

instead of the system $(M(x, y_n))_{n=0,1,\ldots}$ with

$$M(x,y) = \frac{1}{|x|}\sum_{k=0}^{\infty}\left(\frac{|y|}{|x|}\right)^k P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right),\; x \in \overline{A_{\mathrm{ext}}},\; y \in \mathcal{Y} \subset A_{\mathrm{int}}, \tag{16}$$

provided that $\mathcal{Y}$ is a fundamental system in A_{int} with $\kappa = \sup_{y\in\mathcal{Y}}|y| < \alpha$, and the coefficients σ_k, $\sigma_k \neq 0$ for $k = 0, 1, \ldots$, have to be chosen in such a way that

$$\sum_{k=0}^{\infty}(2k+1)|\sigma_k|\left(\frac{\kappa}{\alpha}\right)^k < \infty. \tag{17}$$

Example 5.4. *Suppose that $\mathcal{Y} = (y_n)_{n=0,1,\ldots}$ is a fundamental system in A_{int} with $\kappa = \sup_{y\in\mathcal{Y}}|y| < \alpha$. Let $K(x, y_n)$ be given by (15) (with coefficients σ_k, $\sigma_k \neq 0$ for $k = 0, 1, \ldots$, satisfying the condition (17)). Then*

$$\left(K(x, y_n)\Big|_{x\in A}\right)_{n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(A)$.

The proof of the completeness for the system $(K(\cdot, y_n))_{n=0,1,\ldots}$ immediately follows from the completeness of the spherical harmonics.

Remark 5.5. Of numerical significance are series expansions (15) with explicit (i.e. elementary) representation (as, for example, in the case of (16)).

Example 5.6. *Let y_0 be a fixed point in A_{int}. Denote by $P_n^{y_0}(x)$ the expression given by*

$$\left(\frac{\partial}{\partial y_0}\right)^{\beta} K(x, y_0)\Bigg|_{[\beta]=n}, \quad n = 0, 1, \ldots$$

$$\left(\beta : \text{multiindex}, [\beta] = \beta_1 + \beta_2 + \beta_3, \ \left(\frac{\partial}{\partial y_0}\right)^{\beta} = \frac{\partial^{[\beta]}}{\partial y_1^{\beta_1} \partial y_2^{\beta_2} \partial y_3^{\beta_3}}\Bigg|_{y_0}\right).$$

Then

$$\left(\left(\frac{\partial}{\partial y_0}\right)^{\beta} K(x, y_0)\Bigg|_{[\beta]=n}\Bigg|_{x\in A}\right)_{n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(A)$.

The proof follows from Maxwell's representation theorem. (*cf. e.g.* [40 p. 44])

Applying the Kelvin transform with respect to the sphere A with radius α around the origin (*cf. e.g.* [35]) Example 5.4 leads us to systems

$$\left(\overline{K}(x, \overline{y}_n)\Big|_{x\in\overline{A_{\text{ext}}}}\right)_{n=0,1,\ldots}$$

with

$$\overline{K}(x, \overline{y}) = \sum_{k=0}^{\infty} \frac{2k+1}{4\pi\alpha^2}\sigma_k \left(\frac{\alpha^2}{|x||\overline{y}|}\right)^{k+1} P_k\left(\frac{x}{|x|}\cdot\frac{\overline{y}}{|\overline{y}|}\right), \ x \in \overline{A_{\text{ext}}}, \overline{y} \in \overline{\mathcal{Y}} \subset \overline{A_{\text{ext}}},$$

where $\overline{\mathcal{Y}} = (\overline{y}_n)_{n=0,1,\ldots}$ is the point system generated by $\mathcal{Y}$ by letting $\overline{y}_n = \frac{\alpha^2}{|y_n|^2} y_n$, $n = 0, 1, \ldots$ (thereby assuming $0 \notin \mathcal{Y}$).

Remark 5.7. Note that our assumptions imply the estimate

$$\sum_{k=0}^{\infty} (2k+1)|\sigma_k| \left(\frac{\alpha}{\overline{\kappa}}\right)^k < \infty, \tag{18}$$

where $\overline{\kappa}$ *is given by*

$$\overline{\kappa} = \inf_{\overline{y}\in\overline{\mathcal{Y}}} |\overline{y}| > \alpha.$$

Example 5.8. *Suppose that* $\overline{\mathcal{Y}} = (\overline{y}_n)_{n=0,1,\ldots}$ *is given as described above. Let* $\overline{K}(x, \overline{y}_n)$ *be given as above (with coefficients* σ_k, $\sigma_k \neq 0$ *for* $k = 0, 1, \ldots$, *satisfying (17)). Then*

$$\left(\overline{K}(x, \overline{y}_n)\big|_{x\in A}\right)_{n=0,1,\ldots}$$

is a linearly independent complete system in $\mathrm{L}^2(A)$.

Typical examples of this type are known from harmonic spline and wavelet theory [16, 29, 38, 39] and geodetic implementations (see [27] and the references therein). We only mention:

(i) *Abel-Poisson kernel:*

$$\sigma_k = 1, \quad k = 0, 1, \ldots .$$

The kernel reads as follows:

$$\overline{K}(x, \overline{y}) = \frac{1}{4\pi} \frac{|x|^2|\overline{y}|^2 - \alpha^2}{(L(x, \overline{y}))^{3/2}}, \quad x \in \overline{A_{\text{ext}}},\ \overline{y} \in \overline{\mathcal{Y}} \subset \overline{A_{\text{ext}}},$$

where we have introduced the abbreviation

$$L(x, \overline{y}) = |x|^2|\overline{y}|^2 - 2\alpha^2 x \cdot \overline{y} + \alpha^4.$$

(ii) *Singularity kernel:*

$$\sigma_k = \frac{2}{2k+1}, \quad k = 0, 1, \ldots .$$

The kernel is given by

$$\overline{K}(x, \overline{y}) = \frac{1}{2\pi} \frac{1}{(L(x, \overline{y}))^{1/2}}, \quad x \in \overline{A_{\text{ext}}}, \overline{y} \in \overline{\mathcal{Y}} \subset \overline{A_{\text{ext}}}.$$

(iii) *Logarithmic kernel:*

$$\sigma_k = \frac{1}{(k+1)(2k+1)}, \quad k = 0, 1, \ldots .$$

Now we have

$$\overline{K}(x, \overline{y}) = \frac{1}{4\pi\alpha^2} \log\left(\frac{\alpha^2 - x \cdot \overline{y} + (L(x, \overline{y}))^{1/2}}{|x||\overline{y}| + x \cdot \overline{y}}\right), x \in \overline{A_{\text{ext}}}, \overline{y} \in \overline{\mathcal{Y}} \subset \overline{A_{\text{ext}}}.$$

Remark 5.9. Choosing (instead of (17) and (18)) σ_k, $\sigma_k \neq 0$ *for* $k = 0, 1, \ldots$, *in such a way that*

$$\sum_{k=0}^{\infty} (2k+1)|\sigma_k| < \infty$$

i.e. $(|\sigma_k|^{-1/2})_{k=0,1,\ldots}$ *is assumed to be summable (in the sense of [40] p. 88),* κ *and* $\overline{\kappa}$ *are allowed to satisfy* $\kappa \leq \alpha$ *and* $\overline{\kappa} \geq \alpha$*, respectively.*

An equivalent statement to the completeness of a system $(\Phi_n)_{n=0,1,\ldots}$ in the space $\mathrm{L}^2(A)$ is the *closure* (see *e.g.* [43, p. 191] for the proof of the equivalence): A system $(\Phi_n)_{n=0,1,\ldots}$, $\Phi_n \in \mathrm{L}^2(A)$, $n = 0, 1, \ldots$ is called *closed* in $\mathrm{L}^2(A)$ if, for a given function $\Phi \in \mathrm{L}^2(A)$ and arbitrary $\varepsilon > 0$, there exist an integer $N(= N(\varepsilon))$ and constants $a_0, \ldots, a_N$ such that

$$\left(\int_A \left|\Phi(x) - \sum_{n=0}^{N} a_n \Phi_n(x)\right|^2 \mathrm{d}\omega(x)\right)^{1/2} \leq \varepsilon.$$

The closure particularly means that any $\Phi \in C^{(0)}(A)$ can be approximated by a member of the span of $(\Phi_n)_{n=0,1,\ldots}$ in the sense of the L^2–metric on A.

The step from approximation on the sphere A to approximation in the outer space A_{ext} can be performed by the following theorem (*cf.* [9, 10, 37]):

THEOREM 5.10 *Let* $\mathcal{K}$ *be a (not necessarily compact) subset of the closed outer space* $\overline{A_{\text{ext}}}$ *with* $\operatorname{dist}(\mathcal{K}, A) \geq \tau > 0$. *Then there exists a positive constant* $C = C(\mathcal{K}, A)$ *such that*

$$\sup_{x\in\mathcal{K}} \left|\tilde{\Phi}(x) - \tilde{\Psi}(x)\right| \le C \left(\int_A \Big(\Phi(y) - \Psi(y)\Big)^2 d\omega(y) \right)^{1/2}$$

for all functions $\tilde{\Phi}, \tilde{\Psi}$ *of class* $\mathrm{Pot}^{(0)}(\overline{A_{\mathrm{ext}}})$ *with* $\tilde{\Phi}|A = \Phi, \tilde{\Psi}|A = \Psi$.

Proof. Theorem 5.10 is easily verified by application of the Poisson integral formula

$$\tilde{\Phi}(x) - \tilde{\Psi}(x) = \int_A P(x,y)\Big(\Phi(y) - \Psi(y)\Big) d\omega(y),$$

where $P(x,y)$ denotes the Abel-Poisson kernel (see *e.g.* [35, pp. 240ff.]). Put

$$C = C(\mathcal{K}, A) = \sup_{x\in\mathcal{K}} \left(\int_A \Big(P(x,y)\Big)^2 d\omega(y) \right)^{1/2}. \tag{19}$$

Then, for each $x \in \mathcal{K}$, the Cauchy-Schwarz inequality yields

$$\Big(\tilde{\Phi}(x) - \tilde{\Psi}(x)\Big)^2 \le C^2 \int_A \Big(\Phi(y) - \Psi(y)\Big)^2 d\omega(y). \tag{20}$$

This is the desired result. □

Let $\tilde{\Phi} \in \mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$ with $\tilde{\Phi}|A = \Phi$. If now $(\tilde{\Phi}_n)_{n=0,1,\ldots} \subset \mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$ is given such that $\tilde{\Phi}_n|A = \Phi_n, n = 0,1,\ldots$, forms a complete system in $\mathrm{L}^2(A)$, then for every value $\varepsilon > 0$ there exist an integer $N(= N(\varepsilon))$ and coefficients $a_0, \ldots, a_N$ such that

$$\sup_{x\in\mathcal{K}} \left|\tilde{\Phi}(x) - \sum_{n=0}^{N} a_n \tilde{\Phi}_n(x)\right| \le C \left(\int_A \left(\Phi(y) - \sum_{n=0}^{N} a_n \Phi_n(y)\right)^2 d\omega(y) \right)^{1/2} \le C\varepsilon$$

for each subset $\mathcal{K} \subset A_{\mathrm{ext}}$ with $\mathrm{dist}(\mathcal{K}, A) \ge \tau > 0$, where C in general depends on the choice of $\mathcal{K}$ and A. In other words, given $\tilde{\Phi} \in \mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$ with $\tilde{\Phi}|A = \Phi$, the L^2-approximation of the function Φ on the surface A implies uniform approximation of $\tilde{\Phi}$ by the system $(\tilde{\Phi}_n)_{n=0,1,\ldots}$ on each subset $\mathcal{K}$ of A_{ext} with positive distance to A.

The system $(\tilde{\Phi}_n)$ is a ‘basis system’ (more precisely: *scalar basis system*) in the following sense: Each $\tilde{\Phi} \in \mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$ can be approximated, uniformly on subsets of A_{ext} with positive distance to A, by finite linear combinations of $(\tilde{\Phi}_n)_{n=0,1,\ldots} \subset \mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$, *i.e.* for every function $\tilde{\Phi} \in \mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$ there exists a member $U \in \mathrm{span}_{n=0,1,\ldots}(\tilde{\Phi}_n)$ in ε-accuracy (with respect to the $C^{(0)}(\mathcal{K})$–norm) on every set $\mathcal{K}$ with $\mathrm{dist}(\mathcal{K}, A) \ge \tau > 0$.

As a particular case we mention

$$\sup_{x\in\overline{\Sigma_{\mathrm{ext}}}} \left|\tilde{\Phi}(x) - \sum_{n=0}^{N} a_n \tilde{\Phi}_n(x)\right| \le C\varepsilon.$$

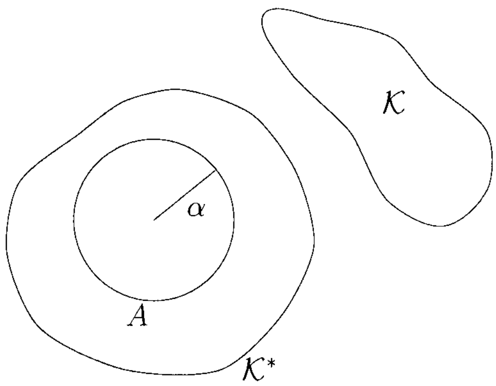

Figure 8. Illustration of the sets A, $\mathcal{K}$ and $\mathcal{K}^*$

6. Vectorial/tensorial approximation

Let $\mathcal{K}$ be a compact subset of A_{ext}. Since A_{ext} is assumed to be an open set, $\mathcal{K}$ has a positive distance to the boundary A. Hence, there exists a regular surface $\mathcal{K}^*$ with $\mathcal{K} \subset \mathcal{K}^*_{\text{ext}}$ and $\overline{\mathcal{K}^*_{\text{ext}}} \subset A_{\text{ext}}$ (*cf.* Figure 8).

In order to prove a basic theorem about vectorial and tensorial approximation, we have to estimate $\sup_{x\in\mathcal{K}} \left|\left(\nabla\tilde{\Phi}\right)(x)\right|$ and $\sup_{x\in\mathcal{K}} \left|\left(\nabla^{(2)}\tilde{\Phi}\right)(x)\right|$, respectively. Let $\mathcal{D} \in \{\nabla, \nabla^{(2)}\}$ be a differential operator. Given $\tilde{\Phi} \in \text{Pot}^{(0)}\left(\overline{A_{\text{ext}}}\right)$, we have

$$\sup_{x\in\mathcal{K}} \left|\left(\mathcal{D}\tilde{\Phi}\right)(x)\right| = \sup_{x\in\mathcal{K}} \left|\mathcal{D}_x \int_{\mathcal{K}^*} \tilde{\Phi}(y) \frac{\partial}{\partial n(y)} G^*(x,y)\,\mathrm{d}\omega(y)\right|, \tag{21}$$

where G^* denotes Green's function for the scalar Dirichlet problem (*cf. e.g.* [35]) in $\mathcal{K}^*_{\text{ext}}$. Consequently, it follows that

$$\sup_{x\in\mathcal{K}} \left|\left(\mathcal{D}\tilde{\Phi}\right)(x)\right| \leq \sup_{x\in\mathcal{K}^*} \left|\tilde{\Phi}(x)\right| \sup_{x\in\mathcal{K}} \int_{\mathcal{K}^*} \left|\mathcal{D}_x \frac{\partial}{\partial n(y)} G^*(x,y)\right| \mathrm{d}\omega(y). \tag{22}$$

Setting

$$C^* = C^*(\mathcal{K}, \mathcal{K}^*, \mathcal{D}) = \sup_{x\in\mathcal{K}} \int_{\mathcal{K}^*} \left|\mathcal{D}_x \frac{\partial}{\partial n(y)} G^*(x,y)\right| \mathrm{d}\omega(y), \tag{23}$$

we find

$$\sup_{x\in\mathcal{K}} \left|\left(\mathcal{D}\tilde{\Phi}\right)(x)\right| \leq C^* \sup_{x\in\mathcal{K}^*} \left|\tilde{\Phi}(x)\right|. \tag{24}$$

Since $\mathcal{K}$ is a compact set in A_{ext}, we are able to deduce the following statement:

THEOREM 6.1.
(i) Each scalar basis system $\left(\tilde{\Phi}_n\right)_{n=0,1,\ldots}$*, i.e. each subsystem* $\left(\tilde{\Phi}_n\right)_{n=0,1,\ldots}$ *of* $\text{Pot}^{(0)}\left(\overline{A_{\text{ext}}}\right)$*, where* $\left(\tilde{\Phi}_n|A\right)_{n=0,1,\ldots}$ *is complete in* $\mathrm{L}^2(A)$*, implies a 'vectorial basis system' in the following*

sense: For $v \in \mathrm{pot}(A_{\mathrm{ext}})$*, there exists an approximation by a finite linear combination of vector fields* $\left(\nabla\tilde{\Phi}_n\right)_{n=0,1,\ldots}$*, uniformly on compact subsets of* A_{ext}.

(ii) *Each scalar basis system* $\left(\tilde{\Phi}_n\right)_{n=0,1,\ldots}$*, i.e. each subsystem* $\left(\tilde{\Phi}_n\right)_{n=0,1,\ldots}$ *of* $\mathrm{Pot}^{(0)}\left(\overline{A_{\mathrm{ext}}}\right)$*, where* $\left(\tilde{\Phi}_n|A\right)_{n=0,1,\ldots}$ *is complete in* $\mathrm{L}^2(A)$*, implies a 'tensorial basis system' in the following sense: For* $\mathbf{v} \in \mathbf{pot}(A_{\mathrm{ext}})$*, there exists an approximation by a finite linear combination of tensor fields* $\left(\nabla^{(2)}\tilde{\Phi}_n\right)_{n=0,1,\ldots}$*, uniformly on compact subsets of* A_{ext}.

Proof. We shall only prove the second part, for the first part can be proved analogously. Suppose that $\mathbf{v}$ is of class $\mathbf{pot}(A_{\mathrm{ext}})$ and $\mathcal{K}$ is a compact subset of A_{ext}. Then there exists a function $V \in \mathrm{Pot}(A_{\mathrm{ext}})$ such that $\mathbf{v}|\mathcal{K} = \nabla^{(2)}V|K = (\nabla\otimes\nabla)V|\mathcal{K}$. Now, for arbitrary $\varepsilon > 0$, we have an integer $N(= N(\varepsilon))$ and coefficients $a_0, \ldots, a_N$ such that

$$\sup_{x\in\mathcal{K}^*}\left|V(x) - \sum_{n=0}^{N} a_n\tilde{\Phi}_n(x)\right| \le \varepsilon.$$

In connection with (24) this gives us

$$\sup_{x\in\mathcal{K}}\left|\mathbf{v}(x) - \sum_{n=0}^{N} a_n\left((\nabla\otimes\nabla)\,\tilde{\Phi}_n\right)(x)\right| \le C^* \sup_{x\in\mathcal{K}^*}\left|V(x) - \sum_{n=0}^{N} a_n\tilde{\Phi}_n(x)\right| \le C^*\varepsilon.$$

This is the desired result. □

7. C-closure

We discuss the relations between the spaces $\mathrm{pot}(A_{\mathrm{ext}})|\overline{\Sigma_{\mathrm{ext}}}$ and $\mathrm{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$. Of course, we have

$$\mathrm{pot}(A_{\mathrm{ext}})|\overline{\Sigma_{\mathrm{ext}}} \subset \mathrm{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right). \tag{25}$$

The inclusion is, in fact, strict: choose $y \in A_{\mathrm{ext}}\backslash\overline{\Sigma_{\mathrm{ext}}}$, then the field

$$x \mapsto \nabla_x \frac{1}{|x-y|}, \qquad x \neq y, \tag{26}$$

is an element of class $\mathrm{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$, but it is obvious that the vector field is not an element of $\mathrm{pot}(A_{\mathrm{ext}})|\overline{\Sigma_{\mathrm{ext}}}$. Hence,

$$\mathrm{pot}(A_{\mathrm{ext}})|\overline{\Sigma_{\mathrm{ext}}} \neq \mathrm{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right). \tag{27}$$

However, we are able to prove the following *closure theorem* (see [36]):

THEOREM 7.1. *The space* $\mathrm{pot}(A_{\mathrm{ext}})|\overline{\Sigma_{\mathrm{ext}}}$ *is a dense subset of* $\mathrm{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$ *with respect to* $\|\cdot\|_{c^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)}$*, i.e. for any given value* $\varepsilon > 0$ *and any element* $v \in \mathrm{pot}^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)$ *there exists a field* $u \in \mathrm{pot}(A_{\mathrm{ext}})|\overline{\Sigma_{\mathrm{ext}}}$ *such that*

$$\|v-u\|_{c^{(0)}\left(\overline{\Sigma_{\mathrm{ext}}}\right)} \le \varepsilon,$$

i.e.

$$\sup_{x \in \overline{\Sigma_{\rm ext}}} |v(x) - u(x)| \leq \varepsilon.$$

The closure theorem (Theorem 7.1) enables us to derive the following approximation theorem:

THEOREM 7.2. *Let* $(\tilde{\Phi}_n)_{n=0,1,\dots}$ *be a system of functions* $\tilde{\Phi}_n \in \mathrm{Pot}^{(0)}\left(\overline{A_{\rm ext}}\right)$, $n = 0, 1, \dots,$ *such that* $(\tilde{\Phi}_n|A)_{n=0,1,\dots}$ *is complete in* $\mathrm{L}^2(A)$. *Then, every function* $v \in \mathrm{pot}^{(0)}\left(\overline{\Sigma_{\rm ext}}\right)$ *can be approximated in the metric* $\|\cdot\|_{c^{(0)}\left(\overline{\Sigma_{\rm ext}}\right)}$ *by a finite linear combination of the gradient fields* $(\nabla\tilde{\Phi}_n)_{n=0,1,\dots}$, *i.e. for given* $\varepsilon > 0$ *and* $v \in \mathrm{pot}^{(0)}\left(\overline{\Sigma_{\rm ext}}\right)$, *there exist an integer* $N(= N(\varepsilon))$ *and coefficients* $a_0, \dots, a_N$ *such that*

$$\sup_{x \in \overline{\Sigma_{\rm ext}}} \left| v(x) - \sum_{n=0}^{N} a_n \left(\nabla\tilde{\Phi}_n\right)(x) \right| \leq \varepsilon. \tag{28}$$

Proof. In comparison to Theorem 7.1 it remains to prove that any continuous linear functional F on $\mathrm{pot}^{(0)}\left(\overline{\Sigma_{\rm ext}}\right)$ satisfying $F\left(\nabla\tilde{\Phi}_n|\overline{\Sigma_{\rm ext}}\right) = 0$ for $n = 0, 1, \dots,$ is zero on the set $\mathrm{pot}(A_{\rm ext})|\overline{\Sigma_{\rm ext}}$, since this implies that $\mathrm{span}_{n=0,1,\dots}(\nabla\tilde{\Phi}_n|\overline{\Sigma_{\rm ext}})$ is dense in $\mathrm{pot}(A_{\rm ext})|\overline{\Sigma_{\rm ext}}$ with respect to $\|\cdot\|_{c^{(0)}\left(\overline{\Sigma_{\rm ext}}\right)}$ according to a theorem in *e.g.* [44].

Let u be a vector field of class $\mathrm{pot}(A_{\rm ext})$. Then we know that there exists a function $U \in \mathrm{Pot}(A_{\rm ext})$ with $u = \nabla U$. Since $(\tilde{\Phi}_n)_{n=0,1,\dots}$ is assumed to be a scalar basis system in $\overline{A_{\rm ext}}$, the function U can be approximated by finite linear combinations U_N of $(\tilde{\Phi}_n)_{n=0,1,\dots}$, i.e. $U_N \to U$ on each compact subset $\mathcal{K}$ of $A_{\rm ext}$. A result given in [41, p. 190] shows that any partial derivative of U_N tends to the corresponding partial derivative of U uniformly on each compact subset $\mathcal{K}$ of $A_{\rm ext}$. We consider, in particular, the second-order derivatives and a bounded neighbourhood of Σ. Then, by application of the mean value theorem of multidimensional analysis, $\nabla U_N \to \nabla U$ in the norm $\|\cdot\|_{c^{(0)}\left(\overline{\Sigma_{\rm ext}}\right)}$. In accordance with the assumption $F\left(\nabla U_N|\overline{\Sigma_{\rm ext}}\right) = 0$. Hence, the continuity of F gives us

$$F(u|\overline{\Sigma_{\rm ext}}) = F\left(\nabla U|\overline{\Sigma_{\rm ext}}\right) = \lim_{N\to\infty} F\left(\nabla U_N|\overline{\Sigma_{\rm ext}}\right) = 0,$$

as required. □

Hence, the external gravitational field v of the Earth admits a uniform approximation by gradient fields of scalar basis systems of class $\mathrm{Pot}^{(0)}\left(\overline{A_{\rm ext}}\right)$ on and outside the Earth's surface.

From an extended version of the Helly Theorem (see [45]) we are able to derive the following corollaries, which play an important role in hi-lo SST of determining the Earth's gravitational field from a finite set of GPS-SST data.

COROLLARY 7.3. *(hi-lo SST) Let the assumptions of Theorem 7.2 be fulfilled. Let* χ *be a finite subset of* $\Gamma \subset \Sigma_{\rm ext}$ *satisfying (8). Then, for given* $\varepsilon > 0$ *and* $v \in \mathrm{pot}^{(0)}\left(\overline{\Sigma_{\rm ext}}\right)$, *there exist an integer* $N(= N(\varepsilon))$ *and coefficients* $a_0, \dots, a_N$ *such that*

$$\sup_{x \in \overline{\Sigma_{\rm ext}}} \left| v(x) - \sum_{n=0}^{N} a_n \left(\nabla\tilde{\Phi}_n\right)(x) \right| \leq \varepsilon$$

and

$$v(x) = \sum_{n=0}^{N} a_n \left(\nabla\tilde{\Phi}_n\right)(x), \quad x \in \chi.$$

COROLLARY 7.4. *(lo-lo SST) Let the assumptions of Corollary 7.3 be fulfilled. Then, for given* $\varepsilon > 0$ *and* $v \in \mathrm{pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$*, there exist an integer* $N(= N(\varepsilon))$ *and coefficients* $a_0, \dots, a_N$ *such that*

$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} \left| v(x) - \sum_{n=0}^{N} a_n \left(\nabla \tilde{\Phi}_n \right)(x) \right| \leq \varepsilon \tag{29}$$

and

$$h(x) \cdot (v(x) - v(x + h(x))) = \sum_{n=0}^{N} a_n h(x) \cdot \left(\left(\nabla \tilde{\Phi}_n \right)(x) - \left(\nabla \tilde{\Phi}_n \right)(x + h(x)) \right), \tag{30}$$

$x \in \chi$*, where* h *is the intersatellite distance.*

COROLLARY 7.5. *(SGG) Under the assumptions of Corollary 7.3 we have*

$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} \left| v(x) - \sum_{n=0}^{N} a_n \left(\nabla \tilde{\Phi}_n \right)(x) \right| \leq \varepsilon$$

and

$$(-x) \cdot (\nabla v(x)(-x)) = \sum_{n=0}^{N} a_n \left((-x) \cdot \nabla_x\right) \left((-x) \cdot \nabla_x\right) \tilde{\Phi}_n(x),$$

$x \in \chi$.

COROLLARY 7.6 *(Combined SST/SGG) Let the assumptions of Corollary 7.3 be fulfilled. Then, for given* $\varepsilon > 0$ *and* $v \in \mathrm{pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$*, there exist an integer* $N(= N(\varepsilon))$ *and coefficients* $a_0, \dots, a_N$ *such that*

$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} \left| v(x) - \sum_{n=0}^{N} a_n \left(\nabla \tilde{\Phi}_n \right)(x) \right| \leq \varepsilon$$

and

$$(-x) \cdot v(x) = \sum_{n=0}^{N} a_n (-x) \cdot \nabla \tilde{\Phi}_n(x),$$

$x \in \mathcal{X}_1$,

$$h(x) \cdot (v(x) - v(x + h(x))) = \sum_{n=0}^{N} a_n h(x) \cdot \left(\left(\nabla \tilde{\Phi}_n \right)(x) - \left(\nabla \tilde{\Phi}_n \right)(x + h(x)) \right),$$

$x \in \mathcal{X}_2$*, and*

$$(-x) \cdot (\nabla v(x)(-x)) = \sum_{n=0}^{N} a_n \left((-x) \cdot \nabla_x\right) \left((-x) \cdot \nabla_x\right) \tilde{\Phi}_n(x),$$

$x \in \mathcal{X}_3$*, where* h *is the intersatellite distance and* $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 = \mathcal{X}$.

In other words, the geopotential field admits an approximation (in ε-accuracy with respect to the uniform topology on $\overline{\Sigma_{\text{ext}}}$) consistent with combined scalar hi-lo SST, lo-lo SST, and SGG data.

8. Multiscale approximation

Of course, there still remain two essential problems, namely the choice of the basis system $\{\tilde{\Phi}_n\}_{n=0,1,\dots}$ and the appropriate strategy of determining the coefficients in the linear combination consistent with the satellite data.

(i) Concerning the *choice of the basis system* a particular role is played by the system of outer harmonics. The polynomial structure has tremendous advantages. In fact, outer harmonics are classical means of modelling the long-wavelength parts of the Earth's gravitational field. But, according to the uncertainty principle (*cf.* [16, 46, 47]), the ideal frequency localization implies no space localization. Outer harmonics as non-space-localizing structures need a uniformly dense coverage of data everywhere. Local changes are not treatable locally; they affect all constituting elements, *i.e.* the whole table of Fourier (orthogonal) coefficients. The critical point, besides numerical problems, is that equidistributed material of sufficiently small data width must be handled by a trial system of non-space localizing functions. In the opinion of the authors, therefore, the numerical use of outer harmonics is limited for modelling satellite data containing medium-to-short-wavelengths features. As a matter of fact, the uncertainty principle in constructive approximation tells us that there exists a hierarchy of the scalar basis functions (mentioned in Section 5) characterized by Figure 15. What we really need for the future satellite scenario are more and more space-localizing basis systems in order to model medium-to-short-wavelengths features of the Earth's gravitational potential. In this respect it should be mentioned that satellite-to-satellite tracking (hi-lo SST) may be considered to be the interface of outer harmonics and kernel functions, whereas satellite gravity gradiometry (SGG) represents the interface of medium-to-strongly space-localizing kernel functions, which seems to be equivalent to the interface of bandlimited and non-bandlimited kernel functions (see also Figure 15).

(ii) Many methods concerned with *numerical procedures for determining linear combinations approximating the Earth's gravitational field* are available in the literature. Probably best known are collocational, least-squares, or Galerkin methods. Usually, large linear systems must be solved to guarantee a sufficient accuracy. However, satellite methods provide us with extremely huge numbers of data. Standard mathematical theory and numerical methods are not at all adequate for the handling of data systems with a structure such as this, because these methods are simply not adapted to the specific character and number of the spaceborne data. They quickly reach their capacity limit, even on very powerful computers. In the opinion of the authors a reconstruction of the gravitational field requires careful (multi)scale analysis, fast solution techniques, and a proper stabilization of the solution by regularization. Regularization can be formulated as a multiresolution analysis (for other strategies see *e.g.* [16, 29, 30] and the references therein). Economical multiscale recovering of the Earth's gravitational field is provided by fast wavelet mechanisms (tree algorithms and pyramid schemata) thereby avoiding completely the solution of any linear system. Essential numerical components of multiscale approximation from spaceborne data have been described in [16]. Future results on gravitational field determination should concentrate on combined models (see Corollary 7.6), where expansions (linear combinations) in terms of outer harmonics have to be combined with more and more space-localizing kernel functions. Even for local approximation the philosophy of the authors developed from the uncertainty principle is the following three-step procedure for modelling the data on the orbit of the satellite. First an outer harmonic approach should be used to model the global trends, *i.e.* the low-wavelengths part. In a second step band-limited wavelets showing moderate space localizing phenomena may be taken for the medium-frequency band of the Earth's gravitational potential. Finally, the third step consists of non-bandlimited wavelet approximation to analyse the fine structure, *i.e.* short-wavelengths phenomena for local areas within a global concept (*cf.* [16]). The numerical background of this approach is justified by the results of this paper. However, for purposes of 'downward continuation' of satellite data we have to regularize the three-step solution by use of (non-bandlimited) Tikhonov regularization (see [16, 17]) or (bandlimited) truncated singular-value decomposition (see [16, 17, 39]).

Current Status	Future Concepts	
Potential differences, geoid heights, satellite altimetry etc.	satellite - to satellite tracking observables	satellite gravity gradiometry observables
Fourier (orthogonal) expansion by outer harmonics	one-/multilevel kernel function approximation	

The idea of *multiscale regularization by (bandlimited) truncated singular value decomposition* is illustrated (in a heuristic way) by the following prodecure: Suppose that there are known from the field $v \in \mathrm{pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$, *i.e.* the gradient field $v = \nabla V$ of the actual Earth's gravitational potential V, the set of scalar values $\{L_1^{N_J}, \ldots L_{N_J}^{N_J}\}$ corresponding to the observational functionals $L_{x_i^{N_J}}$ given by

$$L_i^{N_J} = L_{x_i^{N_J}}(V) = \begin{cases} ((-x)\cdot\nabla_x)\, V(x)\big|_{x=x_i^{N_J}} & \text{(SST)} \\ ((-x)\cdot\nabla_x)\,((-x)\cdot\nabla_x)\, V(x)\big|_{x=x_i^{N_J}} & \text{(SGG)} \end{cases} \tag{31}$$

for some set $\chi^{N_J} = \{x_1^{N_J}, \ldots, x_{N_J}^{N_J}\}$ on the spherical orbit Γ with radius $\gamma, \gamma > \sup_{x\in\Sigma}|x|$, of the LEO, where the sets χ^{N_j}, $j = J_0, \ldots, J$, are given suitably in hierarchical way by setting

$$\begin{aligned} \chi^{N_{J_0}} &= \left\{x_1^{N_{J_0}}, \ldots, x_{N_{J_0}}^{N_{J_0}}\right\}, \\ \subset \chi^{N_{J_0+1}} &= \left\{x_1^{N_{J_0+1}}, \ldots, x_{N_{J_0+1}}^{N_{J_0+1}}\right\} \\ &\cdots\cdots \\ \subset \chi^{N_J} &= \left\{x_1^{N_J}, \ldots, x_{N_J}^{N_J}\right\} \end{aligned} \tag{32}$$

(without loss of generality we may assume that $x_i^{N_j} = x_i^{N_{j+1}}$ for $i = 1, \dots, N_j$, $j = J_0, \dots, J-1$). The developments in the preceding sections of this paper show that, corresponding to $v \in \mathrm{pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$, there exists a field $u = \nabla U$, $U \in \mathrm{Pot}^{(1)}(\overline{A_{\mathrm{ext}}})$, such that $u \mid \overline{\Sigma_{\mathrm{ext}}}$ is in an $(\varepsilon/3)$-neighbourhood to v (understood in the $c^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$-topology) and $L_i^{N_J} = L_{x_i^{N_J}}(U)$, $i = 1, \dots, N_J$. Now, corresponding to the field $u \in \mathrm{pot}^{(0)}(\overline{A_{\mathrm{ext}}})$, there exists in an $(\varepsilon/3)$-neighbourhood to u, *e.g.* a linear combination $w_J = \nabla W_J$ given by

$$W_J = \sum_{i=1}^{N_J} a_i^{N_J} K_J\left(\cdot, x_i^{N_J}\right) \tag{33}$$

with

$$K_J(x, y) = \sum_{k=1}^{\infty} \frac{2k+1}{4\pi\alpha^2} \mathrm{e}^{-k2^{-J}} \left(\frac{\alpha^2}{|x||y|}\right)^{k+1} P_k\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \tag{34}$$

($(x, y) \in \overline{A_{\mathrm{ext}}} \times \overline{A_{\mathrm{ext}}}$, J sufficiently large) such that $L_i^{N_J} = L_{x_i^{N_J}}(W_J)$, $i = 1, \dots, N_J$. Finally, corresponding to $w_J \in \mathrm{pot}^{(0)}(\overline{A_{\mathrm{ext}}})$, there exists in an $(\varepsilon/3)$-neighbourhood to w_J (understood in the $c^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$-topology) a linear combination $w_J^{M_J} = \nabla W_J^{M_J}$ given by

$$W_J^{M_J} = \sum_{i=1}^{N_J} \tilde{a}_i^{N_J} K_J^{M_J}\left(\cdot, x_i^{N_J}\right) \tag{35}$$

with $K_J^{M_J}$ a 'bandlimited variant' of K_J defined by

$$K_J^{M_J}(x, y) = \sum_{k=0}^{M_J} \frac{2k+1}{4\pi\alpha^2} \mathrm{e}^{-k2^{-J}} \left(\frac{\alpha^2}{|x||y|}\right)^{k+1} P_k\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \tag{36}$$

($(x, y) \in \overline{A_{\mathrm{ext}}} \times \overline{A_{\mathrm{ext}}}$, J and M_J sufficiently large) such that $L_i^{N_J} = L_{x_i^{N_J}}(W_J^{M_J})$, $i = 1, \dots, N_J$. In other words, regularization is obtained by use of bandlimited kernel representation. It remains to calculate the coefficients $\tilde{a}_i^{N_J}$, $i = 1, \dots, N_J$.

Obviously, there are two serious difficulties in the aforementioned procedure of finding the approximate linear combination $\nabla W_J^{M_J}$ of v in $\overline{\Sigma_{\mathrm{ext}}}$. First, the approach is non-constructive in the sense that the *a priori* choice of the integers N_J and M_J is unknown. Second, our particular computational interest is not in establishing the linear combination by interpolation (or smoothing in the error affected case) because of the huge amount of satellite data (for more details on interpolation and smoothing by harmonic splines see [16, 30, 31, 38, 40] and the references therein). Therefore, we are required to find a suitable way of multiscale approximation in the sense that v can be approximated sufficiently well by a suitable linear combination of the representation (35) thereby satisfying $L_i^{N_J} \simeq L_{x_i^{N_J}}(W_J^{M_J})$, $i = 1, \dots, N_J$.

An economical and efficient multiscale method for establishing an appropriate linear combination of the scalar 'satellite data function' $x \mapsto L_x(U)$, $x \in \overline{A_{\mathrm{ext}}}$, given by

$$L_x(U) = \begin{cases} ((-x)\cdot\nabla_x)U(x) & \text{(SST)} \\ ((-x)\cdot\nabla_x)((-x)\cdot\nabla_x)U(x) & \text{(SGG)} \end{cases} \tag{37}$$

can be deduced from harmonic wavelet theory (*cf.* [16]). According to this approach we let

$$L_x(U) \simeq L_x\left(W_J^{M_J}\right)$$
$$= \sum_{i=1}^{N_{J_0}} \tilde{a}_i^{N_{J_0}} (\Lambda K)_{J_0}^{M_{J_0}} \left(x, x_i^{N_{J_0}}\right) + \sum_{j=J_0}^{J-1} \sum_{i=1}^{N_j} \tilde{a}_i^{N_j} (\Lambda H)_j^{M_j} \left(x, x_i^{N_j}\right),$$

where

$$(\Lambda K)_j^{M_j}(x,y) = \sum_{k=0}^{M_j} \frac{2k+1}{4\pi\alpha^2} \mathrm{e}^{-k2^{-j}} \Lambda^\wedge(k) \left(\frac{\alpha^2}{|x||y|}\right)^{k+1} P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), \tag{38}$$

$j = J_0, \ldots, J$, and

$$(\Lambda H)_j^{M_j}(x,y) = (\Lambda K)_{j+1}^{M_{j+1}}(x,y) - (\Lambda K)_j^{M_j}(x,y), \quad j = J_0, \ldots, J-1, \tag{39}$$

with $M_{J_0} \le M_{J_0+1} \le \ldots \le M_J$.

The sequence $\{\Lambda^\wedge(k)\}_{k=0,1,\ldots}$ is given by

$$\Lambda^\wedge(k) = \begin{cases} k+1 & \text{(SST)} \\ (k+1)(k+2) & \text{(SGG).} \end{cases} \tag{40}$$

Note that the used kernels are bandlimited counterparts to the Abel-Poisson (kernel) scaling functions (discussed in [40, p. 108]). Following the wavelet theory of [16, pp. 146–350] and assuming (extremely) dense data material, the coefficients $\tilde{a}_i^{N_j}$ may be supposed by Weyl's law (*cf.* [40, p. 166]) to be simply given in the form

$$\tilde{a}_i^{N_J} = \frac{1}{N_J} L_{x_i^{N_J}}(U), \quad i = 1, \ldots, N_J \tag{41}$$

and

$$\tilde{a}_i^{N_j} = \frac{1}{N_j} \int_\Gamma L_x(U)(SH)_j^{M_j}\left(x, x_i^{N_j}\right) \mathrm{d}\omega(x);\ j = J_0, \ldots, J-1;\ i = 1, \ldots, N_j; \tag{42}$$

where $(SH)_j^{M_j}$, $j = J_0, \ldots, J$, is the *Shannon kernel* defined by

$$(SH)_j^{M_j}(x,y) = \sum_{k=0}^{M_j} \frac{2k+1}{4\pi\gamma^2} \left(\frac{\gamma^2}{|x||y|}\right)^{k+1} P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right). \tag{43}$$

Now, we have, for $i = 1, \ldots, N_j$

$$\begin{aligned} &\tilde{a}_i^{N_j} \\ &= \frac{1}{N_j}\int_\Gamma L_x(U)(SH)_j^{M_j}\left(x, x_i^{N_j}\right)\mathrm{d}\omega(x) \\ &= \frac{1}{N_j}\int_\Gamma L_x(U)\int_\Gamma (SH)_j^{M_j}\left(z, x_i^{N_j}\right)(SH)_{j+1}^{M_{j+1}}(z,x)\mathrm{d}\omega(z)\mathrm{d}\omega(x) \\ &\simeq \frac{1}{N_j}\sum_{l=1}^{N_{j+1}} \frac{1}{N_{j+1}}(SH)_j^{M_j}\left(x_i^{N_j}, x_l^{N_{j+1}}\right)\int_\Gamma L_x(U)(SH)_{j+1}^{M_{j+1}}\left(x, x_l^{N_{j+1}}\right)\mathrm{d}\omega(x) \\ &= \frac{1}{N_j}\sum_{l=1}^{N_{j+1}} \tilde{a}_l^{N_{j+1}}(SH)_j^{M_j}\left(x_i^{N_j}, x_l^{N_{j+1}}\right), \end{aligned} \tag{44}$$

where we have again used Weyl's law (see [40])

$$\int_{\Gamma} (SH)_j^{M_j}\left(z, x_i^{N_j}\right)(SH)_{j+1}^{M_{j+1}}(z,x)\,\mathrm{d}\omega(z)$$
$$\simeq \frac{1}{N_{j+1}} \sum_{l=1}^{N_{j+1}} (SH)_j^{M_j}\left(x_l^{N_{j+1}}, x_i^{N_j}\right)(SH)_{j+1}^{M_{j+1}}\left(x_l^{N_{j+1}}, x\right).$$

In conclusion, the satellite data can be simply read in the initial level J and all coefficients $\tilde{a}_i^{N_j}$, $j = J_0, \ldots, J-1$, can be obtained by recursion. Moreover, it should be noted that the sign '$\simeq$' can be replaced by '$=$' if outer harmonic exact integration formulae (see *e.g.* [16, pp. 99–146]) are applied. In conclusion, $W_J^{M_J}$ can be represented in the form

$$W_J^{M_J} = \sum_{i=1}^{N_{J_0}} \tilde{a}_i^{N_{J_0}} K_{J_0}^{M_{J_0}}\left(x, x_i^{N_{J_0}}\right) + \sum_{j=J_0}^{J-1} \sum_{i=1}^{N_j} \tilde{a}_i^{N_j} H_j^{M_j}\left(x, x_i^{N_j}\right), \tag{45}$$

where $K_j^{M_j}$, $H_j^{M_j}$ are given by

$$K_j^{M_j}(x,y) = \sum_{k=0}^{M_j} \frac{2k+1}{4\pi\alpha^2} \mathrm{e}^{-k2^{-j}} \left(\frac{\alpha^2}{|x||y|}\right)^{k+1} P_k\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), \quad j = J_0, \ldots, J, \tag{46}$$

and

$$H_j^{M_j}(x,y) = K_{j+1}^{M_{j+1}}(x,y) - K_j^{M_j}(x,y), \quad j = J_0, \ldots, J-1. \tag{47}$$

The multiscale approach can be interpreted as follows:

$$w_{J_0}^{\mathrm{lo}}(x) = \sum_{i=1}^{N_{J_0}} \tilde{a}_i^{N_{J_0}} \nabla_x K_{J_0}^{M_{J_0}}\left(x, x_i^{N_{J_0}}\right) \tag{48}$$

can be understood as J_0-level low-pass filter of the vector field v in $\overline{\Sigma_{\mathrm{ext}}}$, while

$$w_{J_0}^{\mathrm{ba}}(x) = \sum_{i=1}^{N_{J_0}} \tilde{a}_i^{N_{J_0}} \nabla_x H_{J_0}^{M_{J_0}}\left(x, x_i^{N_{J_0}}\right) \tag{49}$$

is the J_0-level band-pass filter of v that must be added to $w_{J_0}^{\mathrm{lo}}$ to obtain the (J_0+1)–level low pass filter $w_{J_0+1}^{\mathrm{lo}}$ of the vector field v, *i.e.*

$$w_{J_0+1}^{\mathrm{lo}}(x) = w_{J_0}^{\mathrm{lo}}(x) + w_{J_0}^{\mathrm{ba}}(x), \quad x \in \overline{\Sigma_{\mathrm{ext}}}. \tag{50}$$

Adding the (J_0+1)-level band-pass filter $w_{J_0+1}^{\mathrm{ba}}$ of v

$$w_{J_0+1}^{\mathrm{ba}}(x) = \sum_{i=1}^{N_{J_0+1}} \tilde{a}_i^{N_{J_0+1}} \nabla_x H_{J_0+1}^{M_{J_0+1}}\left(x, x_i^{N_{J_0+1}}\right) \tag{51}$$

we obtain the (J_0+2)–level low pass filter $w_{J_0+2}^{\mathrm{lo}}$ of v, etc. By observing this structure we are finally led to the following decomposition and reconstruction scheme:

$$
\begin{array}{ccccccccc}
v & \to & \tilde{a}_i^{N_J} & \to & \tilde{a}_i^{N_{J-1}} & \to & \dots & \to & \tilde{a}_i^{N_{J_0}} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & w_J^{\mathrm{lo}} & & w_{J-1}^{\mathrm{lo}} & & & & w_{J_0}^{\mathrm{lo}}
\end{array}
$$

Decomposition scheme

$$
\begin{array}{ccccccccccccc}
\tilde{a}_i^{N_{J_0}} & & & & \tilde{a}_i^{N_{J_0+1}} & & & & \tilde{a}_i^{N_{J_0+2}} & & & & \\
\downarrow & & & & \downarrow & & & & \downarrow & & & & \\
w_{J_0}^{\mathrm{ba}} & & & & w_{J_0+1}^{\mathrm{ba}} & & & & w_{J_0+2}^{\mathrm{ba}} & & & & \\
 & \searrow & & & & \searrow & & & & \searrow & & & \\
w_{J_0}^{\mathrm{lo}} & \to & + & \to & w_{J_0+1}^{\mathrm{lo}} & \to & + & \to & w_{J_0+2}^{\mathrm{lo}} & \to & + & \to & \dots
\end{array}
$$

Reconstruction scheme

The above decomposition and reconstruction schemata admit canonical techniques of wavelet thresholding (*cf.* [48]) if the data are affected by errors and statistical *a priori* information is available.

EGM96 is a common model for the Earth's gravitational potential. The multiresolution analysis 'looks at' the Earth's gravitational potential through a microscope, whose resolution gets finer and finer. Thus it associates to the gravitational potential and its radial derivatives a sequence of smoothed versions, labelled by the scale parameter. This aspect is illustrated by the Figures 9 to 14 for the (bandlimited) EGM96 model. For the computations of the figures the potential U and its radial derivatives have been calculated on a finite point grid $\mathcal{X}^{N_j}$ as a simulation of real measurements.

The observables of interest are

$$L_x(U) = ((-x) \cdot \nabla_x)\, U(x)$$

(SST, Figures 9 and 10) evaluated at an orbit with 400 km height,

$$L_x(U) = ((-x) \cdot \nabla_x)\, ((-x) \cdot \nabla_x)\, U(x)$$

(SGG, Figures 11 and 12) at an altitude of 200 km, and U itself at the surface of the Earth. The pyramid scheme, discussed above, allows the calculation of approximations $W_j^{M_j}$; $j = J_0, \dots, J$; to U, such that $L_x(W_j^{M_j})$ approximates $L_x(U)$ in both choices of L. The approximation $W_j^{M_j}$ is defined by (35) where the coefficients $\{\tilde{a}_i^{N_j}\}_{i=1,\dots,N_J;\, j=J_0,\dots,J}$ have been determined by use of the pyramid scheme formula (44).

The left columns of the figures show $L_x(W_j^{M_j})$ and $W_j^{M_j}$, respectively, for $j = 3, \dots, 8$. The right columns illustrate the scale steps, which are given by (*cf.*(45))

$$\sum_{i=1}^{N_j} \tilde{a}_i^{N_j} H_j^{M_j}(x, x_i^{N_j}); \quad j = 3, \dots, 7;$$

in case of the potential, and

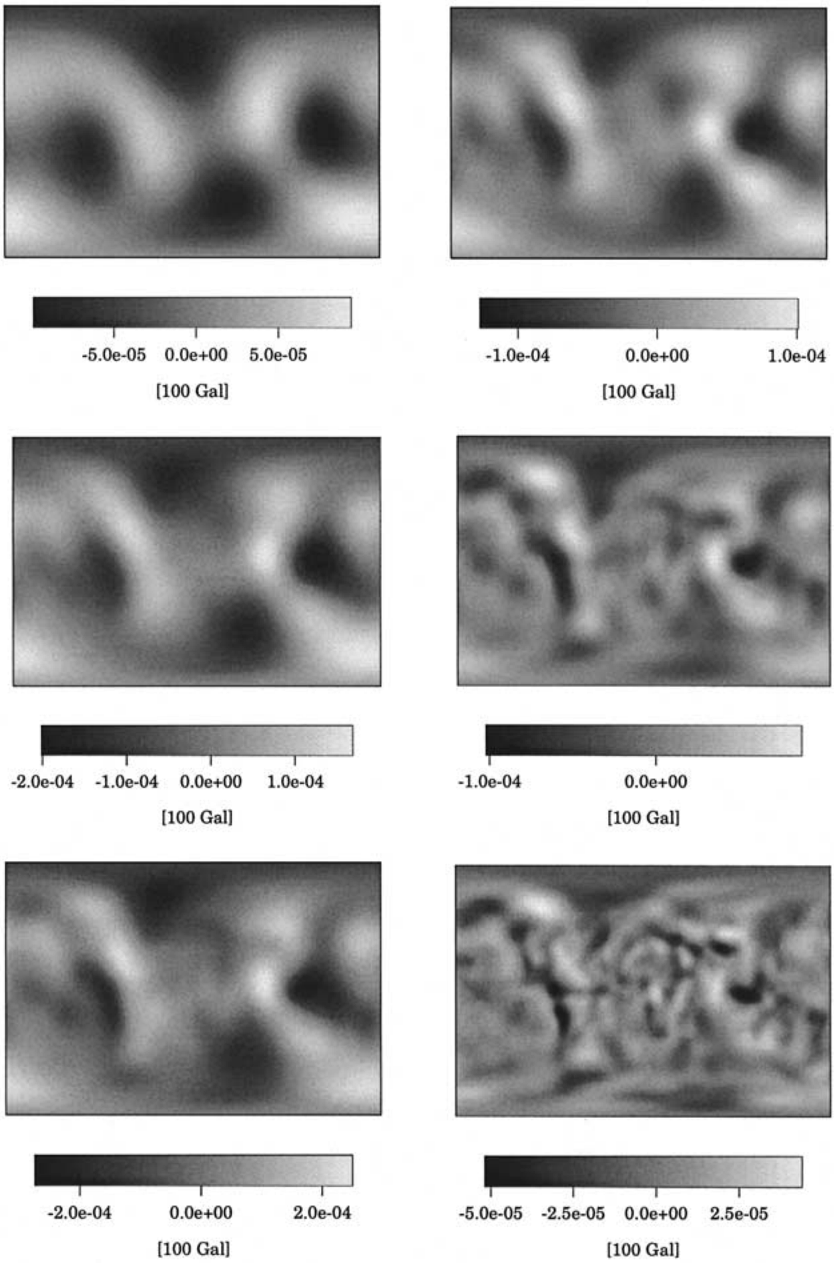

Figure 9. Multiscale representation of the first radial derivative of EGM96 at height 400 km (CHAMP concept) in bandlimited scale spaces (*left*) and detail spaces (*right*); scales 3 (*top*) to 5 (*bottom*)

$$\sum_{i=1}^{N_j} \tilde{a}_i^{N_j} (\Lambda H)_j^{M_j} (x, x_i^{N_j}); \quad j = 3, \ldots, 7;$$

in the case of satellite data.

Summarizing the philosophy of this paper we are finally led to the scheme in Figure 15.

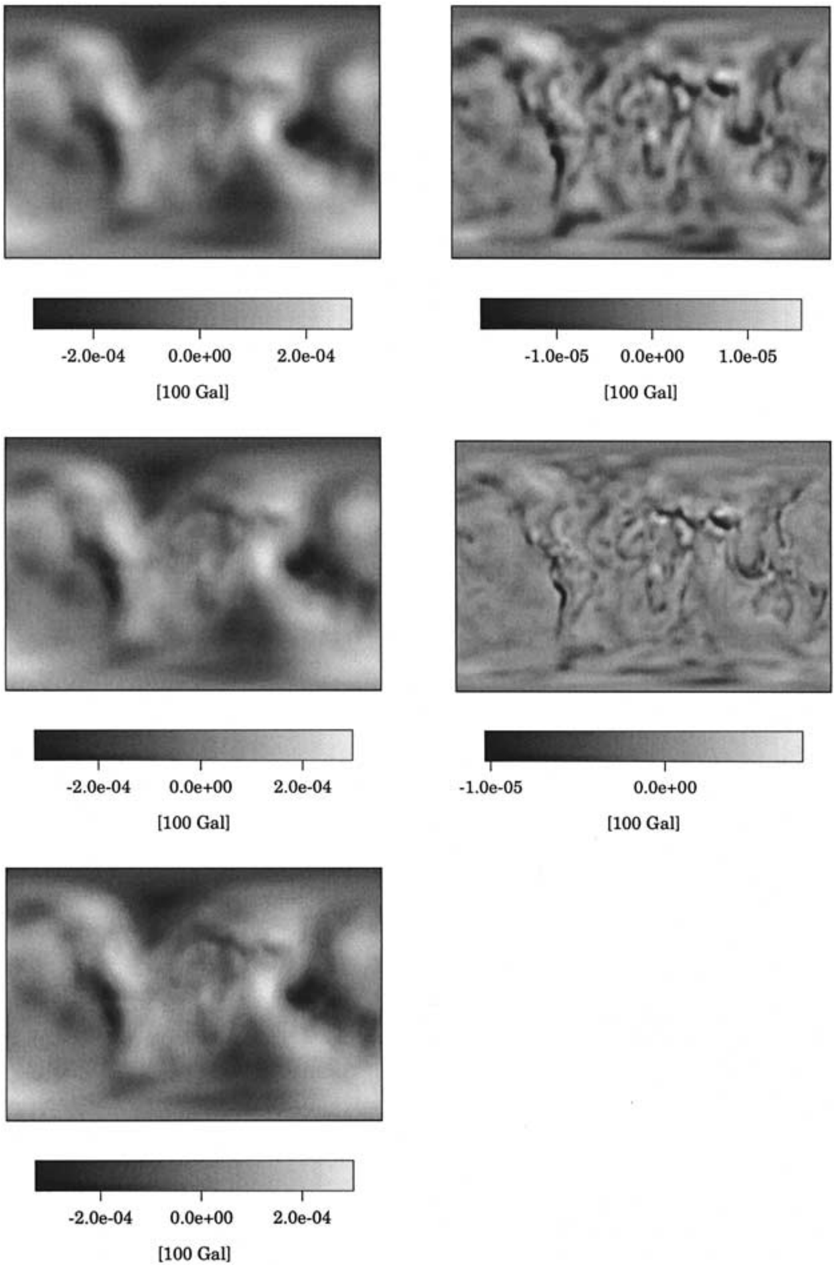

Figure 10. Multiscale representation of the first radial derivative of EGM96 at height 400 km (CHAMP concept) in bandlimited scale spaces (*left*) and detail spaces (*right*); scales 6 (*top*) to 8 (*bottom*)

9. Gravity-field applications

The knowledge of the gravitational field of the Earth is of great importance for many applications in geosciences and industry from which we only mention six significant examples (*cf. e.g.* [22, 23]):

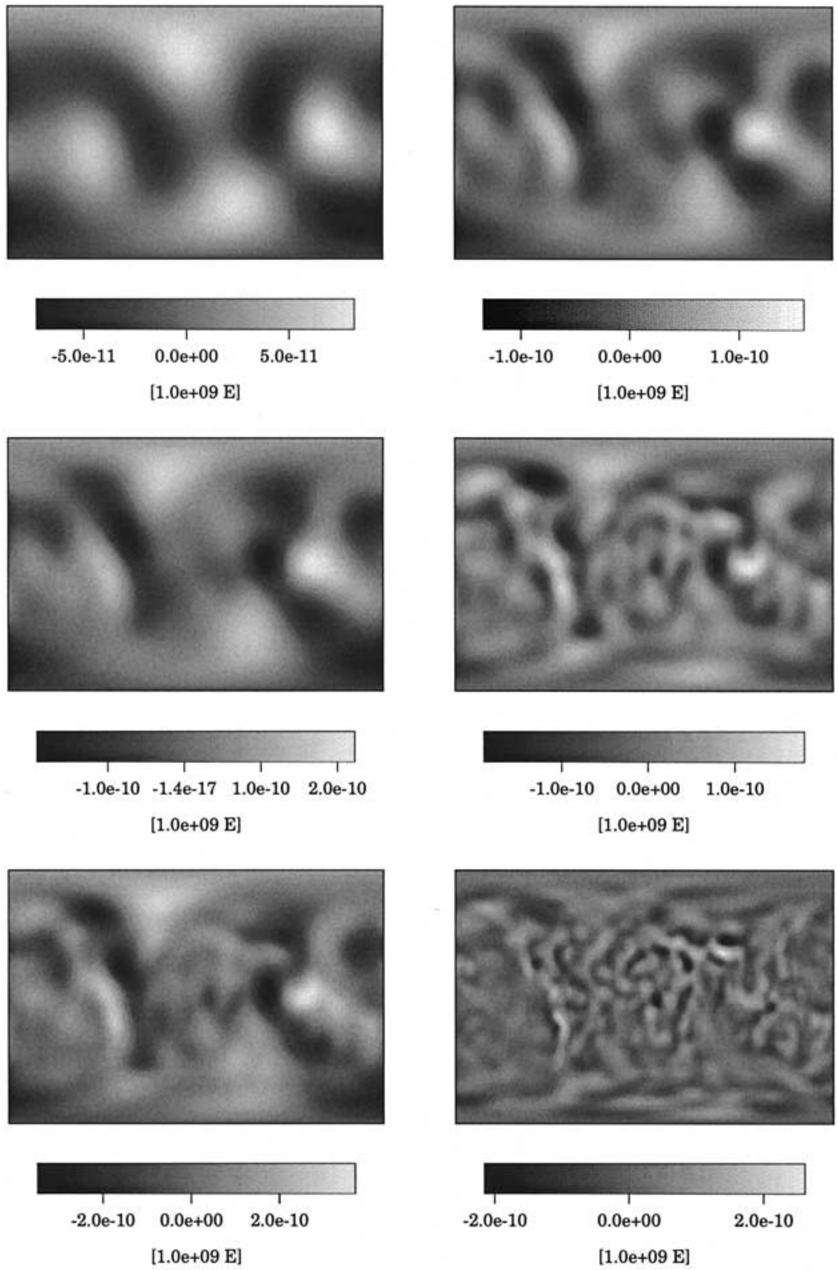

Figure 11. Multiscale representation of the second radial derivative of EGM96 at height 200 km (GOCE concept) in bandlimited scale spaces (*left*) and detail spaces (*right*); scales 3 (*top*) to 5 (*bottom*)

Satellite orbits. For any positioning from space the uncertainty in the orbit of the spacecraft is the limiting factor. The future spaceborne techniques will eliminate basically all gravitational uncertainties in satellite orbits.

Solid-Earth physics. The gravity anomaly field derivable from future satellite observations has its origin mainly in mass inhomogenities of the continental and oceanic lithosphere. To-

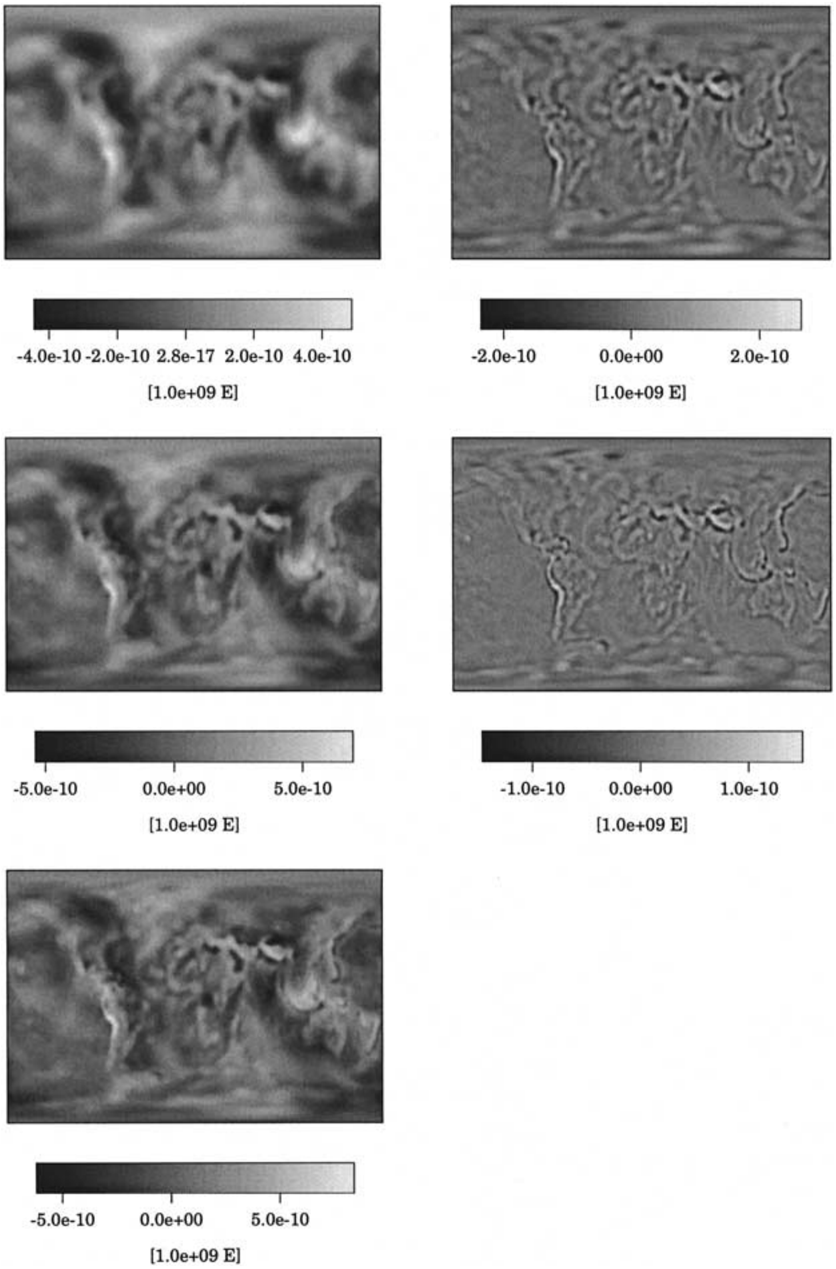

Figure 12. Multiscale representation of the second radial derivative of EGM96 at height 200 km (GOCE concept) in bandlimited scale spaces (*left*) and detail spaces (*right*); scales 6 (*top*) to 8 (*bottom*)

gether with height information and regional tomography, a much deeper understanding of tectonic processes should be obtainable.

Physical oceanography. The future altimeter satellites in combination with a precise geoid will deliver global dynamic ocean topography. From it global surface circulation and its variations in time can be computed resulting in a completely new dimension of ocean modelling. Circulation allows the determination of transport processes of *e.g.* polluted material.

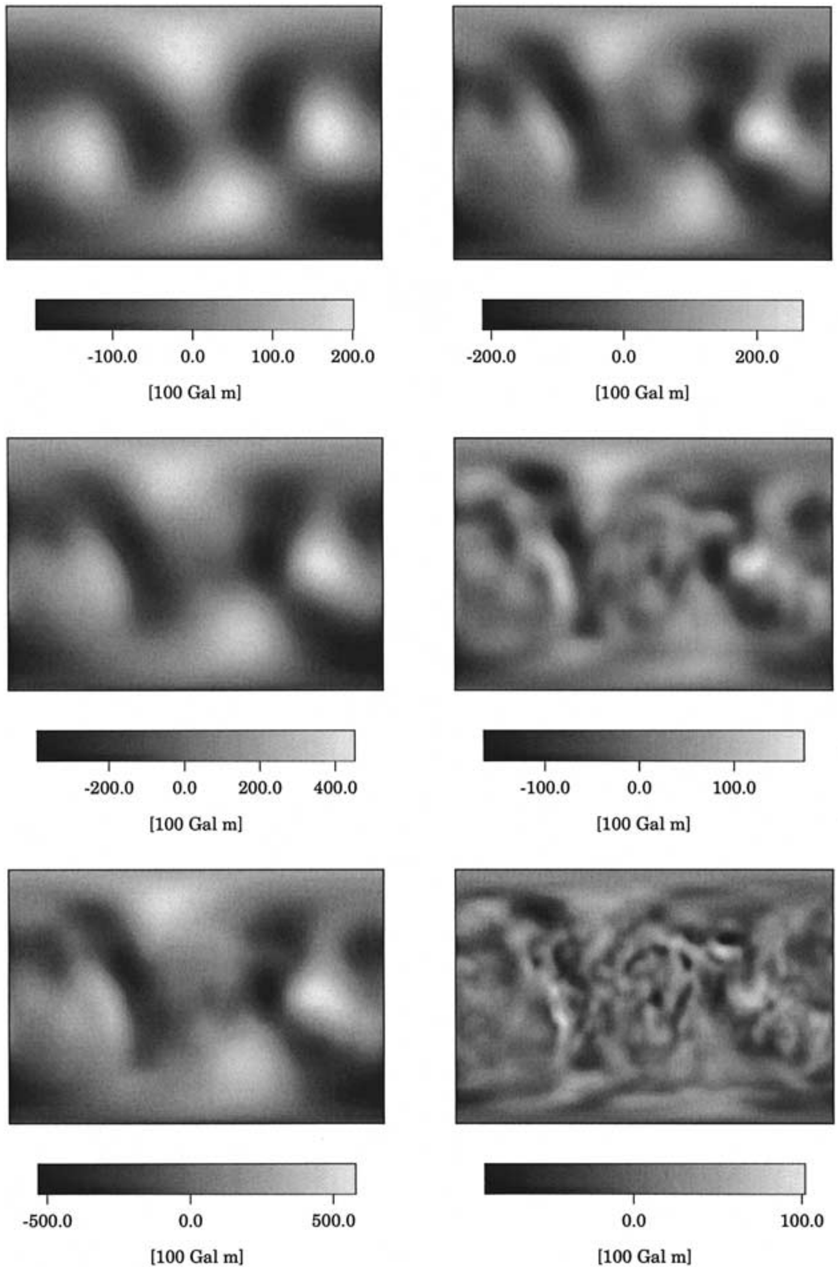

Figure 13. Multiscale representation of EGM96 at height 0 km in bandlimited scale spaces (*left*) and detail spaces (*right*); scales 3 (*top*) to 5 (*bottom*)

Earth system. There is a growing awareness of global environmental problems (for example, the CO_2-question, the rapid decrease of rain forests, global sea level changes, etc.). What is the role of the future airborne methods and satellite missions in this context? They do not tell us the reasons for physical processes, but it is essential to bring the phenomena into one system (*e.g.* to make sea-level records comparable in different parts of the world). In other words, the geoid is viewed as an almost static reference for many rapidly changing processes

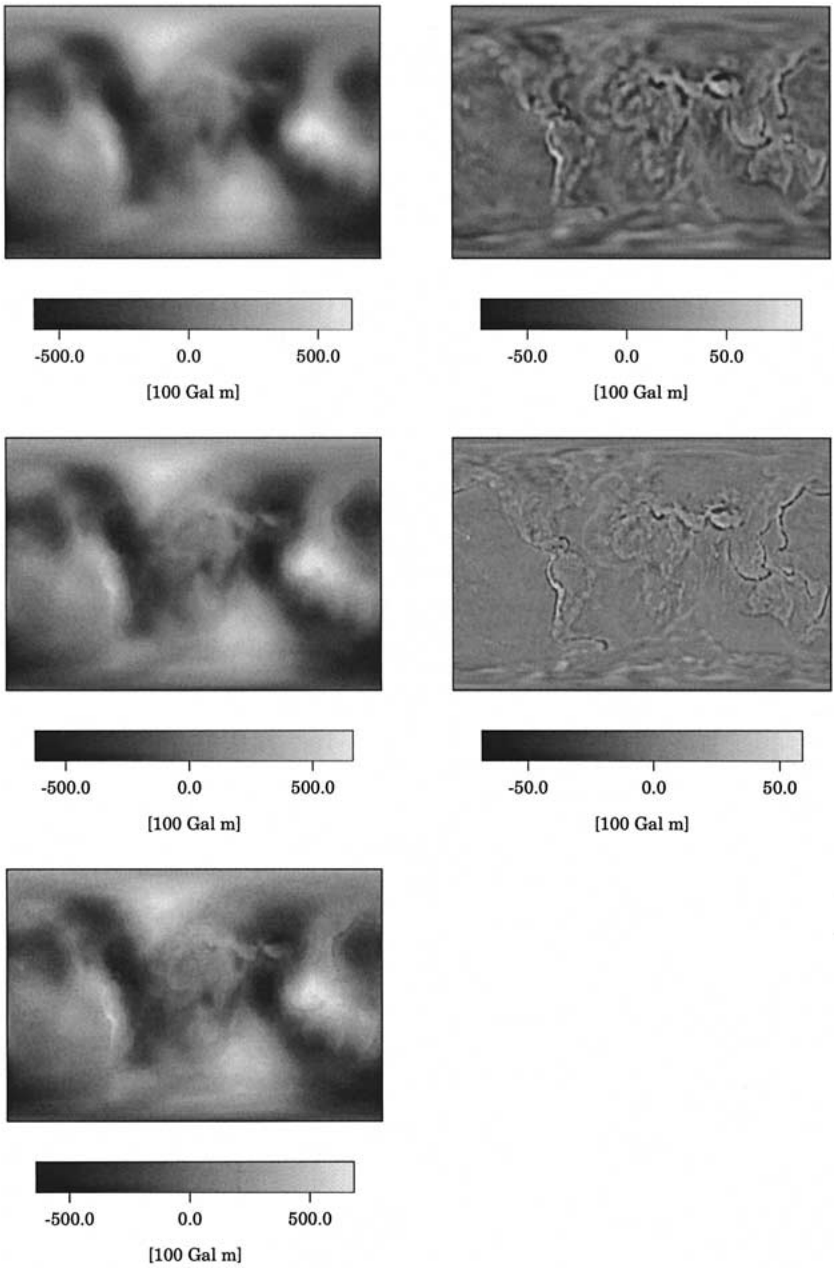

Figure 14. Multiscale representation of EGM96 in bandlimited scale spaces (*left*) and detail spaces (*right*) at height 0 km; scales 6 (*top*) to 8 (*bottom*)

and at the same time as a 'frozen picture' of tectonic processes that evolved over geological time spans.

Geodesy and civil engineering. Accurate heights are needed for civil constructions, mapping, etc. They are obtained by levelling, a very time-consuming and expensive procedure. Nowadays geometric heights can be obtained fast and efficiently from space positioning (for example, GPS and/or GLONASS). The geometric heights are convertible to levelled heights

approx. method	**Fourier**	**splines/wavelets**		**wavelets**
	orthogonal approximation		non-orthogonal approximation	
approx. structure			bandlimited/non-bandlimited	
	polynomials		**kernels**	
	zooming-out		zooming-in	
localization	**increasing frequency localization**, **decreasing frequency localization**			
	decreasing space localization, **increasing space localization**			
	increasing correlation		decreasing correlation	
data structure	**equidistributed**	weakly	**irregular**	strongly
	linear system	linear system/num. integ.		numerical integration
resolution	**long**	**medium**		**short**
		wavelengths		

Figure 15. Survey

by subtracting the precise geoid, which is implied by a high resolution gravitational potential. To be more specific, in those areas where good gravity information is available already, the future data information will eliminate all medium and long-wavelength distortions in unsurveyed areas. GPS and/or GLONASS together with the planned explorer satellite missions for the past 2000 time frame will provide high quality height information at global scale.

Exploration geophysics and prospecting. Airborne gravity measurements have usually been used together with aeromagnetic surveys, but the poor precision of airborne gravity measurements has hindered a wider use of this type of measurements. Strong improvements can be expected from the future scenario. Airborne gravity, of course, has a great advantage because measurements of the gravity field are not restricted to certain areas. Furthermore, knowledge of regional geologic structures can easily be gained by means of airborne data. For purposes of exploration, however, the determination of the absolute gravity field is of little significance as well as gravity anomalies of dimension very much greater than the gravity anomalies caused by *e.g.* the oil and gas structures. The fundamental interest in gravitational methods in exploration is based on the measurements of small variations.

10. Concluding remarks

The missions CHAMP (2000), GRACE (2002), GOCE (2005) will exploit the old concepts of satellite-to-satellite tracking in the high-low mode, in the low-low mode and of satellite gravity gradiometry, respectively. CHAMP is improving our knowledge of the gravity field by about a factor two in resolution. GRACE aims at monitoring periodically time variations of the long-wavelength part of the gravitational field. The GOCE mission has been proposed to provide the most accurate global and high-resolution snapshot of the gravitational field and its corresponding geoidal surface. All satellite missions are intended for use of a wide range of research and application areas, including ocean circulation, climate change studies, and physics of the interior of the Earth. Various scientific teams in Europe involved in this study have designed the mathematical framework of the simulations and the data management. The simulation results mostly are based on the global use of multipole systems, *i.e.* Fourier expansions in terms of outer harmonics. However, concerning the data evaluation we are confronted with the problem that the mathematical methods actually dominating in practice will not be able to handle the new huge satellite data systems neither theoretically nor numerically, particularly if special attention should be payed to the important aspect of an intensified spatially and temporally regionalizing treatment. Moreover, a multiscale analysis by harmonic wavelets as proposed in this work is necessary, *i.e.* a transformation of the geophysically relevant data into constituent elements which is characterized by three essential features: approximation property, decorrelation capability and fast algorithms. These properties are the key to a multitude of possibilities, particularly data compression and transmission, denoising and selective multiresolution.

In conclusion, the Geomathematics Group at the University of Kaiserslautern has been able to develop new mathematical methods for the evaluation of geodata, based on methods of multiscale analysis, which are able to meet the new demands described above. It is planned that these methods shall be further developed systematically and implemented as a homogeneous software structure. On the basis, the future programme ZOOM IN is intended to offer multiscale models within the desired scales from the global observation of our planet from space up to regional dimensions. In doing so, the users have an instrument in their hands that is classified according to wavelength, frequency, space and time, which results in a better understanding of the interrelations and interactions and a scale-specific observation of the system Earth.

In the near future, the Geomathematics Group wants to concentrate on the research areas of the spatial-temporal multiscale analysis of the gravitational field, magnetic field, sea-surface variations and density variations. On the medium term, the application to high-precision geoid, ocean-circulation, processes in the lithosphere and geoprospection will be accounted for. These areas must be dealt with in parallel, which guarantees synergies with respect to methodical know-how, data processing and software structure. However, it is still more important that an integrated concept be developed from the beginning which is capable of handling data of different sources and origin and different observables in common. Examples are the planned connection of the working areas gravitational and magnetic field, as well as the combination of satellite and seismic data.

In the long run, that is after the initial years, the authors are convinced that ZOOM IN will progressively continue this integration in order to develop applications especially for the economically important area of geoprospection. For the research areas mentioned above, we plan an adaptation of the methods, which are at different levels of development, to the needs of the respective scientific or commercial user groups.

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